Efficient Dominance Testing for Unconditional Preferences

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Abstract

We study a dominance relation for comparing outcomes based on unconditional qualitative preferences and compare it with its unconditional counterparts for TCP-nets and their variants. Dominance testing based on this relation can be carried out in polynomial time by evaluating the satisfiability of a logic formula.

Introduction

Representing and reasoning about preferences is the subject of much recent work in AI (Brafman and Domshlak 2009). CP-nets (Boutilier et al. 2004), TCP-nets (Brafman, Domshlak, and Shimony 2006) and their extensions (Wilson 2004b; 2004a) capture qualitative intra-variable preferences and relative importance over a set of variables. Dominance testing for these languages has been shown to be PSPACE-complete (Goldsmith et al. 2008) based on the *ceteris paribus* ("all else being equal") interpretation of preferences.

We consider *TUP-nets*, an unconditional fragment of TCP-nets. We introduce a dominance relation for TUP-nets and compare it with its unconditional counterparts for TCP-nets and their variants. We provide a polynomial time algorithm for dominance testing for TUP-nets. TUP-nets are *not* special cases of already known restrictions of CP-/TCP-nets for which polynomial time dominance testing algorithms exist (Boutilier et al. 2004).

A Language for Unconditional Preferences

Let $\mathcal{X} = \{X_i\}$ be a set of variables, each with a domain D_i. An outcome α is a complete assignment to all the variables, denoted by the tuple $\alpha := \langle\alpha(X_1), \alpha(X_2), \ldots, \alpha(X_m)\rangle$ such that $\alpha(X_i) \in D_i$ for each $X_i \in \mathcal{X}$. We consider a preference language $\mathcal{L}_{TUP}$ for specifying: (a) unconditional intra-variable preferences $\succ_i$ that are strict partial orders (i.e., irreflexive and transitive relations) over D_i for each $X_i \in \mathcal{X}$; and (b) unconditional relative importance preferences $\rhd$ that are strict partial orders over $\mathcal{X}$.

Let $\mathcal{L}_{CP}$, $\mathcal{L}_{TCP}$ and $\mathcal{L}_{Ext}$ denote the preference languages of CP-nets, TCP-nets (an extension of CP-nets) and Wilson's extension to TCP-nets respectively. We note that:

- $\mathcal{L}_{TUP}$ allows the expression of relative importance while $\mathcal{L}_{CP}$ does not; and $\mathcal{L}_{CP}$ allows the expression of conditional intra-variable preferences while $\mathcal{L}_{TUP}$ does not.
- $\mathcal{L}_{TUP}$ is less expressive than $\mathcal{L}_{TCP}$ because it does not allow the expression of conditional preferences.
- When restricted to unconditional preferences, $\mathcal{L}_{TCP} = \mathcal{L}_{TUP}$.
- $\mathcal{L}_{Ext}$ is more expressive than $\mathcal{L}_{TCP}$ (Wilson 2004b; 2004a), and hence, $\mathcal{L}_{TUP}$ as well.

Dominance under *Ceteris Paribus* Semantics

The semantics for dominance testing in the languages $\mathcal{L}_{CP}$, $\mathcal{L}_{TCP}$ and $\mathcal{L}_{Ext}$ were given by Boutilier et al. (Boutilier et al. 2004), Brafman et al. (Brafman, Domshlak, and Shimony 2006) and Wilson (Wilson 2004b; 2004a) respectively. Dominance testing between two outcomes in these languages is cast as a search for a *flipping* sequence of outcomes from one outcome to the other.

Definition 1 (Adapted from (Wilson 2004b; 2004a) for $\mathcal{L}_{TUP}$)**.** *A sequence of outcomes $\gamma_1, \cdots \gamma_n$ is a (worsening) flipping sequence from γ_1 to γ_n iff for $1 \leq i < n$, either*

1. *(V-flip) γ_i differs from γ_{i+1} in the value of exactly one variable X_j, and $\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$, or*
2. *(I-flip) γ_i differs from γ_{i+1} in the value of variables X_j **and** $X_{k_1}, X_{k_2}, \cdots X_{k_l}$, $\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$, and $X_j \rhd X_{k_1}, X_j \rhd X_{k_2}, \cdots, X_j \rhd X_{k_l}$.*

Brafman et al.'s flipping sequence differs from (restricts) the above such that $l = 1$ in any I-flip in a flipping sequence.

Let $\succ^\circ$ and $\succ^\blacksquare$ denote the dominance relation corresponding to the semantics of $\mathcal{L}_{TCP}$ and $\mathcal{L}_{Ext}$ respectively ($\succ^\circ$ also includes the semantics of $\mathcal{L}_{CP}$). Then $\alpha \succ^\blacksquare \beta$ and $\alpha \succ^\circ \beta$ if and only if there exists a flipping sequence from α to β according to Definition 1 (Wilson) and its restriction (Brafman et al.) respectively.

Example 1. *Let $\mathcal{X} = \{X, Y, Z\}$ and $D_X = \{x_1, x_2\}$; $D_Y = \{y_1, y_2\}$; $D_Z = \{z_1, z_2\}$. Suppose that the intra-variable preferences are given by $x_1 \succ_X x_2, y_1 \succ_Y y_2$ and $z_1 \succ_Z z_2$, and the relative importance among the variables is given by $X \rhd Y$ and $X \rhd Z$. If $\alpha = \langle x_1, y_2, z_2\rangle$ and $\beta = \langle x_2, y_1, z_1\rangle$, then $\alpha \not\succ^\circ \beta$ and $\beta \not\succ^\circ \alpha$ but $\alpha \succ^\blacksquare \beta$.*

Dominance testing has been shown to be PSPACE-complete (Goldsmith et al. 2008) for $\mathcal{L}_{CP}$, $\mathcal{L}_{TCP}$ and $\mathcal{L}_{Ext}$.

Dominance Testing for $\mathcal{L}_{TUP}$

We now provide a polynomial time dominance testing approach for $\mathcal{L}_{TUP}$. We proceed by defining a relation $\succeq_i$ (for each variable $X_i \in \mathcal{X}$) that is derived from $\succ_i$.

Definition 2 ($\succeq_i$)**.** $\forall u, v \in D_i : u \succeq_i v \Leftrightarrow u = v \vee u \succ_i v$

Since $\succ_i$ is a strict partial order (irreflexive and transitive), it can be shown that $\succeq_i$ is a preorder (reflexive and transitive). We next define dominance of α over β with respect to $\{\succ_i\}$ and $\rhd$ using a first order logic formula.

Definition 3 (Dominance for Unconditional Preferences)**.** *Given input preferences $\{\succ_i\}$ and $\rhd$, and a pair of outcomes α and β, we say that α **dominates** β (denoted $\alpha \succ^\bullet \beta$) iff:*

$$\exists X_i : \alpha(X_i) \succ_i \beta(X_i) \\ \wedge \forall X_k : (X_k \rhd X_i \vee X_k \sim_\rhd X_i) \Rightarrow \alpha(X_k) \succeq_k \beta(X_k)$$

*where $X_k \sim_\rhd X_i \Leftrightarrow X_k \not\rhd X_i \wedge X_i \not\rhd X_k$, and X_i is called the **witness** of the relation.*

Intuitively, this definition of dominance of α over β (i.e., $\alpha \succ^\bullet \beta$) requires that α is *preferred* to β with respect to at least one variable, namely the witness. Further, it requires that for all variables that are relatively more important than or indifferent to the witness, α is *either equal to or is preferred to* β. In Example 1, $\alpha \succ^\bullet \beta$, with witness X_1.

We list some properties of $\succ^\bullet$ below (see (Santhanam, Basu, and Honavar 2009) for proofs). First, $\succ^\bullet$ is strict, i.e., no outcome is preferred over itself.

Proposition 1 (Irreflexivity of $\succ^\bullet$)**.** $\forall \alpha : \alpha \not\succ^\bullet \alpha$.

We observe that $\succ^\bullet$ is not transitive when $\{\succ_i\}$ and $\rhd$ are arbitrary strict partial orders, as shown by Example 2.

Example 2. *Let $\mathcal{X} = \{X_1, X_2, X_3, X_4\}$, and for each $X_i \in \mathcal{X} : D_i = \{a_i, b_i\}$ with $a_i \succ_i b_i$. Suppose that $X_1 \rhd X_3$ and $X_2 \rhd X_4$. Let $\alpha = \langle a_1, a_2, b_3, b_4\rangle$, $\beta = \langle b_1, a_2, a_3, b_4\rangle$ and $\gamma = \langle b_1, b_2, a_3, a_4\rangle$. Clearly, we have $\alpha \succ^\bullet \beta$ (with X_1 as witness), $\beta \succ^\bullet \gamma$ (with X_2 as witness), but there is no witness for $\alpha \succ^\bullet \gamma$, i.e., $\alpha \not\succ^\bullet \gamma$ according to Definition 3.*

Since transitivity is a necessary condition for rational choice (French 1986), it is interesting to explore whether $\succ^\bullet$ is transitive under certain restrictions. It turns out that when $\succ_i$'s are arbitrary partial orders, $\succ^\bullet$ is transitive if and only if $\rhd$ is restricted to be an *interval order*, a special type of strict partial order.

Definition 4 (Interval Order)**.** *A binary relation $\boldsymbol{R} \subseteq \mathcal{X} \times \mathcal{X}$ is an interval order iff it is irreflexive and satisfies the* Ferrers *axiom (Fishburn 1985): for all $X_i, X_j, X_k, X_l \in \mathcal{X}$, we have: $(X_i \boldsymbol{R} X_j \wedge X_k \boldsymbol{R} X_l) \Rightarrow (X_i \boldsymbol{R} X_l \vee X_k \boldsymbol{R} X_j)$*

In other words, $\rhd$ is an interval order if and only if it has *no restriction* that is isomorphic to the partial order $X_i \rhd X_j \wedge X_k \rhd X_l$ (Fishburn 1985).

Proposition 2 (Transitivity of $\succ^\bullet$)**.** *If $\rhd$ is an interval order, then $\forall \alpha, \beta, \gamma : \alpha \succ^\bullet \beta \wedge \beta \succ^\bullet \gamma \Rightarrow \alpha \succ^\bullet \gamma$.*

Theorem 1. *If intra-variable preferences $\{\succ_i\}$ are partially ordered, then $\succ^\bullet$ is transitive if and only if relative importance $\rhd$ is an interval order.*

Given partially ordered intra-variable preferences, the preceding theorem holds for a wide range of relative importance preferences including total orders, weak orders and semi orders (Fishburn 1985) which are all interval orders.

Dominance testing in $\mathcal{L}_{TUP}$ amounts to evaluating the satisfiability of $\alpha \succ^\bullet \beta$ (Santhanam, Basu, and Honavar 2009), which can be done in $O\big(m^2(m^4 + n^4)\big)$ time, where $m = |\mathcal{X}|$ is number of variables and $n = max_{X_i \in \mathcal{X}} |D_i|$ is size of the domains of variables.

Semantics: Relationship Between $\succ^\circ$, $\succ^\blacksquare$ & $\succ^\bullet$

We investigate the relationship between the semantics $\succ^\circ$, $\succ^\bullet$, and $\succ^\blacksquare$ for the language $\mathcal{L}_{TUP}$. We show that:

a) $\succ^\bullet \subseteq \succ^\blacksquare$

b) $\succ^\bullet = \succ^\blacksquare$ when $\rhd$ is an interval order

c) $(\succ^\bullet)^\star = \succ^\blacksquare$, where $(\succ^\bullet)^\star$ is the transitive closure of $\succ^\bullet$

d) $\succ^\bullet \not\subseteq \succ^\circ$ and $\succ^\circ \not\subseteq \succ^\bullet$ in general; but $\succ^\circ \subseteq \succ^\bullet$ when $\rhd$ is an interval order

Theorem 2. $\succ^\bullet \subseteq \succ^\blacksquare$.

Proof. We will show that $\alpha \succ^\bullet \beta \Rightarrow \alpha \succ^\blacksquare \beta$ for any pair of outcomes α, β.

Suppose that $\alpha \succ^\bullet \beta$ with witness X_i (see Definition 3). Define the sets $L = \{X_l : X_i \rhd X_l\}$, $M = \{X_l : (X_l \rhd X_i \vee X_l \sim_\rhd X_i) \wedge \alpha(X_l) \succ_l \beta(X_l) \wedge X_l \neq X_i\}$, and $M' = \{X_l : (X_l \rhd X_i \vee X_l \sim_\rhd X_i) \wedge \alpha(X_l) = \beta(X_l) \wedge X_l \neq X_i\}$. Clearly, the sets $\{X_i\}$, L, M, M' form a partition of $\mathcal{X}$. Let $X_{t1}, X_{t2}, \ldots X_{tn}$ be an enumeration of M.

We now construct a sequence of outcomes $\gamma_{t1}, \gamma_{t2}, \ldots, \gamma_{tn}$ corresponding to $X_{t1}, X_{t2}, \ldots X_{tn}$ as follows. $\gamma_{t1} = \langle \gamma_{t1}(X_1), \gamma_{t1}(X_2), \ldots \gamma_{t1}(X_m)\rangle$ such that $\gamma_{t1}(X_{t1}) = \alpha(X_{t1})$ and $\forall X_j \in \mathcal{X} - \{X_{t1}\} : \gamma_{t1}(X_j) = \beta(X_j)$. Similarly $\gamma_{ti} = \langle \gamma_{ti}(X_1), \gamma_{ti}(X_2), \ldots \gamma_{ti}(X_m)\rangle$ such that $\gamma_{ti}(X_{ti}) = \alpha(X_{ti})$; and $\forall X_j \in \mathcal{X} - \{X_{ti}\} : \gamma_{ti}(X_j) = \gamma_{ti-1}(X_j)$.

Now, we make use of Definition 1 to compare these outcomes with respect to $\succ^\blacksquare$. $\gamma_{t1} \succ^\blacksquare \beta$ because $\gamma_{t1}(X_{t1}) = \alpha(X_{t1}) \succ_{t1} \beta(X_{t1})$ with γ_{t1} and β being equal in all variables other than X_{t1} (V-flip). Also $\gamma_{ti+1} \succ^\blacksquare \gamma_{ti}$ because $\gamma_{ti+1}(X_{ti}) = \alpha(X_{ti}) \succ_{ti} \gamma_{ti}(X_{ti}) = \beta(X_{ti})$, with γ_{ti+1} and γ_{ti} being equal in variables other than X_{ti}. For the last outcome in this sequence $\gamma_{t1}, \ldots, \gamma_{tn}$, we have $\alpha \succ^\blacksquare \gamma_{tn}$ because $\alpha(X_i) \succ_i \gamma_{tn}(X_i) = \beta(X_i)$ and $\forall X_l \in M \cup M' : \alpha(X_l) = \gamma_{tn}(X_l)$, regardless of the assignments to variables $X_j \in L$ (they are less important than X_i) (I-flip). Hence, $\alpha \succ^\blacksquare \gamma_{tn} \succ^\blacksquare \ldots \succ^\blacksquare \gamma_{t1} \succ^\blacksquare \beta$. By the transitivity of $\succ^\blacksquare$ (Wilson 2004b; 2004a), $\alpha \succ^\blacksquare \beta$. □

We now investigate the other side of the inclusion. We recall Example 2 that is relevant in this context.

Example 2 (continued). *Recall that $\alpha = \langle a_1, a_2, b_3, b_4\rangle$, $\beta = \langle b_1, a_2, a_3, b_4\rangle$ and $\gamma = \langle b_1, b_2, a_3, a_4\rangle$ with $\alpha \succ^\bullet \beta$ (with X_1 as witness), $\beta \succ^\bullet \gamma$ (with X_2 as witness), but $\alpha \not\succ^\bullet \gamma$ according to Definition 3. However, there exists a sequence of flips from α to γ, namely α, β, γ according to Definition 1. Hence, $\alpha \succ^\blacksquare \gamma$.*

This example shows that $\succ^\blacksquare \subseteq \succ^\bullet$ does not hold in general. However, observe that $\succ^\bullet$ holds for each consecutive pair of outcomes in the flipping sequence. Hence, if $\succ^\bullet$ is transitive, we can show that $\succ^\blacksquare \subseteq \succ^\bullet$.

Theorem 3. $\succ^\blacksquare \subseteq \succ^\bullet$ *when $\rhd$ is an interval order.*

Proof. Given a set of intra-variable preferences $\{\succ_i\}$ and relative importance $\rhd$, we show that $\alpha \succ^\blacksquare \beta \Rightarrow \alpha \succ^\bullet \beta$ when $\rhd$ is an interval order.

Let $\alpha \succ^\blacksquare \beta$. According to Definition 1, there exists a set of outcomes $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}, \gamma_n$ such that $\alpha = \gamma_1 \succ^\blacksquare \gamma_2 \succ^\blacksquare \cdots \succ^\blacksquare \gamma_{n-1} \succ^\blacksquare \gamma_n = \beta$ such that for all $1 \leq i < n$ there is either a *V-flip* or an *I-flip* between γ_i and γ_{i+1}.

Case 1: (V-flip) γ_i and γ_{i+1} differ in the value of exactly one variable X_j and $\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$. With X_j as the witness, the first clause in the definition of $\gamma_i \succ^\bullet \gamma_{i+1}$ is satisfied ($\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$). Because $\gamma_i(X_k) = \gamma_{i+1}(X_k)$ for all $X_k \in \mathcal{X} - \{X_j\}$, we have $\forall X_k : (X_k \rhd X_j \vee X_k \sim_\rhd X_j) \Rightarrow \gamma_i(X_k) \succeq_k \gamma_{i+1}(X_k)$ by Definition 2. Therefore, we have $\gamma_i \succ^\bullet \gamma_{i+1}$ with X_j as the witness.

Case 2: (I-flip) γ_i and γ_{i+1} differ in the value of variables X_j and $X_{k_1}, X_{k_2}, \cdots X_{k_l}$, and $X_j \rhd X_{k_1}, X_j \rhd X_{k_2}, \cdots, X_j \rhd X_{k_l}$, such that $\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$. With X_j as the witness, the first clause in the definition of $\gamma_i \succ^\bullet \gamma_{i+1}$ is satisfied ($\gamma_i(X_j) \succ_j \gamma_{i+1}(X_j)$).

By Definition 1, $\gamma_i(X_k) = \gamma_{i+1}(X_k)$ for all $X_k \in \mathcal{X} - \{X_j, X_{k_1}, X_{k_2}, \cdots, X_{k_l}\}$. In particular, $\gamma_i(X_k) = \gamma_{i+1}(X_k)$ for all X_k such that $X_k \rhd X_j \vee X_k \sim_\rhd X_j$, which means that $\forall X_k : (X_k \rhd X_j \vee X_k \sim_\rhd X_j) \Rightarrow \gamma_i(X_k) \succeq_k \gamma_{i+1}(X_k)$ by Definition 2. Therefore, we have $\gamma_i \succ^\bullet \gamma_{i+1}$ with X_j as the witness by Definition 3[1].

From Cases 1 and 2, $\gamma_i \succ^\bullet \gamma_{i+1}$ for every pair of consecutive outcomes γ_i and γ_{i+1}. Using the fact that $\succ^\bullet$ is transitive when $\rhd$ is an interval order (Theorem 1), we have $\alpha \succ^\bullet \beta$ (by Definition 3) when $\rhd$ is an interval order. Hence, $\succ^\blacksquare \subseteq \succ^\bullet$ when $\rhd$ is an interval order. □

The next observation follows from the fact that $\succ^\bullet$ holds for each pair of consecutive outcomes in a flipping sequence supporting $\alpha \succ^\blacksquare \beta$.

Observation 1. $(\succ^\bullet)^\star = \succ^\blacksquare$, *where $(\succ^\bullet)^\star$ is the transitive closure of $\succ^\bullet$.*

Note that this observation holds even when $\rhd$ is not an interval order. However, it does not yield a computationally efficient algorithm for dominance testing in general because computing $(\succ^\bullet)^\star$ is in itself an expensive operation.

We now investigate the relationship between $\succ^\circ$ and $\succ^\bullet$. In Example 2, α, β, γ forms a flipping sequence from γ to α, resulting in $\alpha \succ^\circ \gamma$ (by Brafman et al.'s definition of a flipping sequence). However, $\alpha \not\succ^\bullet \gamma$. $\alpha \succ^\circ \beta$ implies that there exists a flipping sequence from α to β such that $\succ^\bullet$ holds for each pair of consecutive outcomes in the sequence. Hence, it follows that when $\succ^\bullet$ is transitive, $\succ^\circ \subseteq \succ^\bullet$. On the other hand, Example 1 shows that it is possible that $\alpha \succ^\bullet \beta$ but $\alpha \not\succ^\circ \beta$, and hence, the other side of the inclusion does not hold. This leads us to the following observation.

Observation 2. $\succ^\bullet \not\subseteq \succ^\circ$ *and* $\succ^\circ \not\subseteq \succ^\bullet$ *in general; but* $\succ^\circ \subseteq \succ^\bullet$ *when $\rhd$ is an interval order.*

[1] Note that we do not care how γ_i and γ_{i+1} compare w.r.t. variables $\{X_{k_1}, \cdots, X_{k_l}\}$ that are less important than witness X_j.

Concluding Remarks

Dominance testing for conditional preference languages such as CP-nets, TCP-nets and their extensions have been shown to be computationally hard (Goldsmith et al. 2008). Although polynomial time dominance testing algorithms exist for restricted classes of CP-/TCP-nets, there are no known polynomial time dominance testing algorithms for any preference language that allows expression of relative importance of variables. We study one such language, $\mathcal{L}_{TUP}$, an unconditional fragment of $\mathcal{L}_{TCP}$, the language of TCP-nets. Dominance testing in $\mathcal{L}_{TUP}$ amounts to evaluating the satisfiability of a logic formula that can be carried out in polynomial time.

Our results lead to two natural questions that would be interesting to explore: (1) whether dominance testing using a search for flipping sequences can be achieved in polynomial time in the case of unconditional preferences; and (2) whether the existing large body of work on efficient SAT solvers (Zhang and Malik 2002) can be leveraged to perform efficient dominance testing for other more expressive preference languages.

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