

The Rise and Fall of Complex Family Structures: Coalition Formation, Stability, and Power Struggle

Angelina Brilliantova
Moscow State University
cheli231@gmail.com

Anton Pletenev
Moscow State University
aapletenev@yandex.ru

Hadi Hosseini
Rochester Institute of Technology
hhvcs@rit.edu

ABSTRACT

A complex family is a particular case of family structures in certain animal groups that arises from cooperative breeding. Inspired by this phenomenon, we study the problem of complex family formation from a game-theoretical perspective. While these structures may arise in various domains, such as search and rescue and robotics, the standard types of coalitional games (e.g. simple games) do not capture the observed intricacies of such complex formations. In this paper, we propose a characteristic function game for complex family structures and investigate the stability of coalitions and the existence of the core under various conditions. We provide theoretical bounds on the existence of complex families and the size of coalitions. Furthermore, we empirically examine the proposed framework through an ecological case study and show that our results are consistent with the observed family formations, shedding light on the causes of complex family compositions in the past.

KEYWORDS

Coalition formation; cooperative games; biologically-inspired applications

1 INTRODUCTION

In wildlife populations, cooperative breeding encompasses a variety of social structures as an alternative to monogamous pairing. These structures often result in *complex family* formations wherein offsprings receive additional care from group members (the so-called “helpers”) that are not their parents. Such formations are dynamic, self-organized, and capable to aggregate and dissolve in respond to changes in external conditions. A complex family, at its heart, is a heterogeneous multiagent system, consisting of a central pair of agents (breeding animals) and multiple secondary agents (helpers).

In the past few decades, artificial intelligence approaches have been extensively used to model and analyze various domains from developing effective strategies against invading species in biological sustainability [6] and optimizing landscape connectivity for animal migration [41] to providing theoretical foundations in multiagent systems [13, 28, 42]. In particular, game theory has long been the center of attention in many fields to investigate cooperation or competition over resources in natural and artificial systems where multiple intelligent agents interact to achieve certain goals [23, 30, 31, 34, 40]. For example, evolutionary game theory has been widely applied to analyze the conflicting interests of individuals,

leading to the emergence of cooperative behaviours [8]. However, to date, cooperative breeding, one of the most striking demonstration of animal cooperative behaviour, has received little attention and mainly remained the object of interest amongst ecologists and empiricists alike [5].

The most common approach in theoretical studies of animal cooperation is decomposing it into several agents and modeling their interactions as a non-cooperative game, which requires information about the actions (or strategies) of other agents. One major drawback of non-cooperative games is that they require information about interactions between group members, which is usually hard to measure in practice, particularly when dealing with animals.

To address this issue, empirical biologists focus on the net benefits of cooperative groups (or coalitions) using various ecological parameters e.g. survival, the number of offsprings, environmental characteristics [5]. Coalitional game theory uses a coalition as a basic modeling unit and assigns to each coalition a payoff, that could be distributed among its members. Therefore, coalitional game theory provides a solid mathematical framework for analyzing the group formation strategies with respect to *all* agents, and has been extensively applied to various domains in artificial intelligence and multiagent systems [9, 12, 21, 30, 38, 43]. The goal is to provide solution concepts for determining which cooperative groups are stable (e.g. core) and how the benefits of this cooperation should be divided among the participating agents (e.g. Shapley value).

In this paper, we develop a coalitional game for complex family formation with heterogeneous agents. We compute the stability of these coalitions under a variety of conditions, provide empirical simulations, and further discuss their connections to other applied problems in multiagent and cooperative team formations such as search and rescue. In our analysis, we are inspired by a subspecies of Arctic foxes from Mednyi Island (Western Pacific). In contrast to the mainland population, these subspecies form families with several non-breeding female helpers as well as a breeding pair. Such complex structures with non-breeding animals sometimes occur among several other animal species (e.g. meerkats). Thus we are interested in investigating the rise and fall of such phenomena in animal species.

Our goal is to answer the following questions:

- (1) When do stable complex coalitions arise in a population of agents?
- (2) What is the maximum size of a complex social coalition given the characteristics of external environment and intra-specific parameters of a population?
- (3) How do the parameters of the game shift the power balance and importance of the central and secondary agents?

2 THE FAMILY FORMATION GAME

In this section, we introduce a coalitional game for complex family formation. Let $P = \{m, 1, 2, \dots, n\}$ be a set of agents with m indicating an adult breeding male and $N = \{1, 2, 3, \dots, n\}$ denoting a set of adult females. Let $v : 2^P \rightarrow \mathbb{R}^{\geq 0}$ be a *characteristic function* for each coalition $C \subseteq P$. A *coalitional game* for family formation is then denoted by $G = (P, v)$, where $v(\emptyset) = 0$. A *family* is a coalition of agents that occupy a native home range, which can deliver a maximum payoff (capacity) of 1. Each agent makes a choice: to stay with the family or leave from the coalition and migrate.

We assume that utilities are transferable meaning that each coalition can be assigned a single value as its worth, which can then be distributed among its members. This is a common modeling approach in cooperative game theory for analyzing the formation of coalitions including in weighted voting games, constant sum games, etc. [36, 37]. Many of these settings, e.g. models of a decision-making in political institutions (International Monetary Fund, European Union), consider transferable utilities as an abstraction without providing evidence that agents could redistribute the voting outcome [1, 26, 27].

From the biological perspective, evolutionary theory considers reproduction (or gene propagation) as the primary motivation of animals' behavior [15]. Hence the payoff to each agent in the game G is defined as the expected number of viable offspring (survived until they are one year old) during agent's lifetime. Note that an agent makes a choice for the current year, but the utility depends not only on the reproductive success of the current year but the following years as well. Therefore, even a breeding pair doesn't always divide the payoff equally as one of them could die earlier than the other. The expected number of offspring is a function of agent's life expectancy and its reproductive success. Moreover, the *kin selection theory* postulates that an animal could increase the reproductive success of its relatives at a cost to her own survival and reproduction [22, 39]. Therefore, the benefits of the family is sometimes above the individual agents' reproduction chances, and this payoff is enjoyed by all family members, justifying the choice of our model.

2.1 Game Type

Empirical evidence suggests that in many mammal populations with complex families breeding pairs are widespread and successfully bear offspring without "helpers" (additional members in the family). Moreover, the presence of helpers may not improve reproduction success of a breeding group [25]. Hence, complex families may be beneficial not to a breeding pair but to helpers by increasing their chance of survival and future reproduction. In wildlife populations, most of the potential helpers are offspring of the previous years. These animals either disperse and try to form their own family (*migration*) or may habitually stay in the family (*philopatry*).

Henceforth, unlike simple or constant sum games, a family formation game requires a more intricate characteristic function with an additive component, monotonically rising with each new member to a family coalition, and unique payoff subfunctions for migration and philopatry.

2.2 Characteristic Function

We model the characteristic function of the family formation game as a function of the *number of females*, n , *migration parameter*, h , a breeding *pair utilization* of a home range capacity, a , and a diminishing factor for *marginal contribution* of each additional female to a breeding group, $\gamma \geq 0$. We consider three types of coalitions: 1) a lone male: migrant; 2) a group of females without a male: migrants; and 3) one male and at least 1 female: a family.

While in some animal complex families (e.g. lions and hyenas) multiple females breed, in certain others (e.g. foxes and meerkats) it is more common that only one female plays the role of the breeder [4]. In this paper we focus only on the latter case. Non-breeding females in such groups increase the value of the coalition by improving their own survival and/or by increasing cubs survival of the breeding pair. Nevertheless, the marginal contribution of each additional female to a family is often limited by the maximum capacity of a home range, e.g. food sources, shelters, and the maximum number of cubs. Following the common approach in simulating biological, economical, and financial processes [3, 7, 16, 32, 33], we model the marginal contribution of additional females using an exponentially decreasing function.

Given a diminishing factor γ , $y(i) = e^{-\gamma(i-1)}$ is a function of *diminishing value* for i th additional female, where $y(1) = 1$. Therefore, the characteristic function of the family formation game is written as:

$$v(C) = \begin{cases} h & \text{for } C = \{m\}, \\ h \cdot |C| & \text{for } \forall C \subseteq P : m \notin C \\ f(k) = a + (1-a) \cdot \frac{\sum_{i=1}^k y(i) - y(1)}{d - y(1)} & \text{for } \forall C \subseteq P : m \in C \end{cases} \quad (1)$$

where $a \in [0, 1]$, $h \in [0, 0.5]$, $k = |C \setminus m|$ is a number of females in a coalition C , and $d = \lim_{k \rightarrow +\infty} \sum_{i=1}^k y(i) = \frac{1}{1-e^{-\gamma}}$ is the *normalization factor*.

According to $v(C)$, the maximum value (capacity of a home range) of the grand coalition is $\lim_{|P| \rightarrow +\infty} v(|P|) = \lim_{k \rightarrow +\infty} f(k) = 1$; the value of a breeding pair is $v(\{j, m\}) = f(1) = a$, $\forall j \in N$. The value of a coalition of migrant females is $v(C) \in [0, 0.5n]$, and can be more than 1, as migrant females could 'potentially' establish n home ranges, given available resources in the environment.

In the rest of this paper, for simplicity we use the following helper notations:

- $f(k) = 1 - (1-a) \cdot e^{-\gamma(k-1)}$: the value of a coalition of k females and a male (since $\sum_{i=1}^k y(i)$ is a geometric series).
- $t = 1 - f(n) = (1-a) \cdot e^{-\gamma(n-1)}$: the *remaining capacity* of a home range.
- $\frac{f(k) - f(k-1)}{f(k-1) - f(k-2)} = e^{-\gamma}$: the relative contribution of a female compared to the preceding female joining the coalition, $\forall k > 2$.

2.3 Ecological Interpretation

We assume that each agent (male or female) can gain a positive payoff $h \in [0, 0.5]$ by its own by becoming a migrant and looking for a new coalition. Thus, the parameter h represents the expected

benefits an agent gains from migration or the alternative cost of staying in the habitual family, and depends on the availability of vacant home ranges and partners nearby. If $h = 0$ no vacant home ranges/partners exists, so a migrant have zero chance of reproduction during its lifetime; if $h = 0.5$ a migrant can certainly find a vacant range and a partner and benefit from the maximum capacity of the home range.¹ The female mammals rarely disperse in groups which makes female coalitions temporal, consequently breaking into lone migrants [10], thus, for all coalitions that do not include a male there is a linear increase in characteristic function by h . The pair utilization parameter, a , is the payoff of a coalition consisting of a breeding pair (1 male and 1 female). When a approaches 1, a home range can not afford more agents. If a reaches 0, then a breeding pair needs helpers to gain sufficient payoff.²

2.4 Properties of the Family Formation Game

In this section, we briefly explain the formal definitions and solution concepts for our family formation game.

A family formation game $G = (P, v)$ is *monotonic* if $v(T) \geq v(B)$ for all coalitions $B \subseteq T$, and *superadditive* if for any two disjoint coalitions B and T we have $v(B \cup T) \geq v(B) + v(T)$. A *coalition structure* for the game G is a partition $\mathbb{C} = \{C_1, C_2, \dots, C_g\}$ of P into pairwise disjoint coalitions such that $\bigcup_{i=1}^g C_i = P$.

In coalitional games, we are interested in measuring whether agents will be willing to form a grand coalition (coalition of all agents) given that they can internally distribute the payoff associated to the coalition. If such distribution of values is possible and no sub-coalition has incentive to break away from the grand coalition, then we say that the game has a *nonempty core*. A nonempty core guarantees that the coalitional game has *stable* solutions, meaning that no agent or group of agents have an incentive to deviate [17, 36]. However, the core may be empty for some games, or may contain a large set of solutions.

Let $\mathbf{x} = (x_m, x_1, \dots, x_n) \in \mathbb{R}^{(n+1)}$ be the *payoff vector* of the grand coalition such that $(x_m + \sum_{i=1}^n x_i) = v(P)$, and x_m and x_i denote the payoffs to the male and female $i \in N$ respectively.

The condition that \mathbf{x} is in the core of G , i.e. $\mathbf{x} \in \text{Core}(G)$, can be written in the form of the following linear program:

$$\begin{cases} x_m + \sum_{i=1}^n x_i = v(P) = f(n) = 1 - (1-a) \times e^{-r(n-1)} \\ \sum_{i \in C} x_i \geq v(C) \end{cases} \quad \forall C \subseteq P \quad (2)$$

Let $\mathbb{D} = \{D_1, D_2, \dots, D_b\}$ be any set of pairwise disjoint coalitions consisting of all females, $\bigcup_{i=1}^b D_i = N$, and $\mathbb{A}^T = \{A_1, A_2, \dots, A_g\}$ is any set of pairwise disjoint coalitions consisting of females other than members of T , $T \cap \bigcup_{i=1}^g A_i = \emptyset$ and $T \cup \bigcup_{i=1}^g A_i = N$. In our game the male agent has two choices: to form a coalition with at least one female (breeding group) or form a coalition by its own (forcing all players to migrate). Hence all possible coalitional structures could be divided into two subclasses: $\{\{m \cup T\}, \mathbb{A}^T\}$: a set of a coalition of the male and $|T|$ females plus coalitions of

remaining females; and $\{\{m\}, \mathbb{D}\}$: a set of the coalition of the male alone plus coalitions of all females.

2.5 Coalition Structure and the Core

In order to simplify the analysis of the core we split Equation (2) into two linear programs according to those coalitional structure subclasses, and reformulate the conditions for $\mathbf{x} \in \text{Core}(G)$ using Equation (1).

Definition 2.1. A family formation game G has a nonempty core (is stable) if all of the followings hold:

- For a coalition structure with one of the coalitions consisting of a male and at least one female, that is, $\forall T \subseteq N : T \neq \emptyset, \forall A_j \subseteq \mathbb{A}^T$, we have

$$\begin{cases} x_m + \sum_{i=1}^n x_i = f(n) \\ x_m + \sum_{i \in T} x_i \geq v(\{m, T\}) = f(|T|) \\ \sum_{i \in A_j} x_i \geq v(A_j) = |A_j| \cdot h \end{cases}$$

- For a coalition structure with a coalition consisting of a lone male and the other coalitions of any number of females with no male, $\forall D_j \subseteq \mathbb{D}$ we have

$$\begin{cases} x_m + \sum_{i=1}^n x_i = f(n) \\ x_m \geq v(\{m\}) = h \\ \sum_{s \in D_j} x_s \geq v(D_j) = |D_j| \cdot h \end{cases}$$

Note, that G has a nonempty core only if both linear programs in Definition 2.1 hold true simultaneously for at least one payoff vector. If only the second program has nonempty solution, grand coalition collapses and a breeding group of smaller size is formed, as an alternative, with the other females migrating. In the case of empty solution of the second program (but not the first) all players leave the home range. If two programs have empty solutions simultaneously the outcome depends on the relation between parameter h on one side and a and r on the other. In the next section we use Definition 2.1 to solve each of the two programs analytically and get the parameter combinations for existence of the core.

3 STABILITY OF THE COMPLEX FAMILY

We analyze the existence of a nonempty core in the family formation game in relation to the parameters of the game. We focus on games with $n \geq 2$ as the case of $n = 1$ is trivial.

THEOREM 3.1. A family formation game with n, a, γ has a non-empty core only if $h \leq \hat{h}$, where \hat{h} :

$$\hat{h} = \min \left(\frac{t}{d-1}, \frac{1-t}{n+1} \right) \quad (3)$$

PROOF. Consider the first type of coalition structures in Definition 2.1. By using that $x_m + \sum_{i=1}^n x_i = f(n)$, $|T| = k \leq n$, $\sum_{j=1}^g |A_j| = n - k$ and summing up inequalities $\sum_{i \in A_j} x_i \geq |A_j| \cdot h, \forall j \in (1, 2, \dots, g), \forall g \leq$

¹For simplicity, we assume: 1) all migrants receive identical payoff regardless of sex, 2) they can form only a simple breeding pair and not a complex family, and 3) the maximum payoff is shared equally among a pair.

²Such situations may arise in cases of high pressure of intruders, so a pair cannot defend the territory by its own.

$n - k$, we have

$$\begin{cases} x_m + \sum_{i \in T} x_i = f(n) - \sum_{i \notin T} x_i \geq f(k) \\ \sum_{j=1}^g \sum_{i \in A_j} x_i = \sum_{i \notin T} x_i \geq h \sum_{j=1}^g |A_j| = (n-k) \cdot h \end{cases}$$

which can be written as

$$\begin{cases} \sum_{i \notin T} x_i \leq f(n) - f(k) \\ \sum_{i \notin T} x_i \geq (n-k) \cdot h \end{cases}$$

and further simplified to

$$\begin{aligned} (n-k) \cdot h &\leq \sum_{i \notin T} x_i \leq f(n) - f(k) \\ (n-k) \cdot h &\leq f(n) - f(k) \\ h &\leq \frac{f(n) - f(k)}{n-k}, \forall k \in [1, n-1] \end{aligned} \quad (4)$$

Using Equation (1) that $f(k) = a + (1-a) \cdot \frac{\sum_{i=1}^k y(i)-1}{d-1}$ we have

$$\begin{aligned} \frac{f(n) - f(k)}{n-k} &= \frac{1}{n-k} \cdot \frac{1-a}{d-1} \cdot \left(\sum_{i=1}^n y(i) - \sum_{i=1}^k y(i) \right) \\ &= \frac{1-a}{d-1} \cdot \frac{\sum_{i=k+1}^n y(i)}{n-k} \end{aligned} \quad (5)$$

where $\frac{\sum_{i=k+1}^n y(i)}{n-k}$ is the average of elements of series $y(k+1), \dots, y(n)$, and $y(i) = e^{-\gamma(i-1)}$ is decreasing with k for $\gamma > 0$ and $k \geq 1$. Therefore, the average is also decreasing with k and Equation (5) reaches its minimum at $k = \max(k) = n-1$:

$$\begin{aligned} \min \left(\frac{f(n) - f(k)}{n-k} \right) &= f(n) - f(n-1) = \frac{1-a}{d-1} \cdot y(n) \\ &= \frac{1-a}{d-1} \cdot e^{-\gamma(n-1)} \end{aligned}$$

We rewrite Equation (4) as

$$h \leq \frac{1-a}{d-1} \cdot y(n) = \frac{t}{d-1} \quad (6)$$

Inequality (6) implies that the grand coalition will not collapse if the marginal contribution of the last female is more than the migration parameter.

Now we rewrite the conditions of the second type of coalition structures in Definition 2.1.

Summing up inequalities $\sum_{i \in D_j} x_i \geq |D_j| \cdot h$ for $\forall j \in (1, 2, \dots, b); \forall b \leq n$, we have

$$\begin{cases} x_m = f(n) - \sum_{i \in 1}^n x_i \geq h \\ \sum_{j=1}^b \sum_{i \in D_j} x_i = \sum_{i=1}^n x_i \geq \sum_{j=1}^b |D_j| \cdot h = n \cdot h \end{cases}$$

Thus, we end up with

$$\begin{aligned} n \cdot h &\leq \sum_{i \in 1}^n x_i \leq f(n) - h \\ h &\leq \frac{f(n)}{n+1} = \frac{1 - (1-a) \cdot e^{-\gamma(n-1)}}{n+1} = \frac{1-t}{n+1} \end{aligned} \quad (7)$$

Inequality (7) implies that the grand coalition will not collapse if the payoff of the grand coalition divided by its size is more than the migration parameter. Together Equations (6) and (7) imply the conditions of Theorem 3.1. \square

Simply put, Theorem 3.1 sets the conditions for the game parameters that are necessary for the existence of the core.

PROPOSITION 3.2. *Characteristic function $v(C)$ for the game G with nonempty core is monotonic for $h \leq a$ and superadditive for $h \leq \frac{a}{2}$. Superadditivity of $v(C)$ is a sufficient condition for a nonempty core.*

PROOF. Let's find conditions for monotonicity and superadditivity of characteristic function $v(C)$. According to Equation (1), for $v(C) = h \cdot |C|$ the characteristic function is linear, implying monotonicity and superadditivity. And it is monotonic for $v(C) = f(k)$. Therefore, the break of monotonicity could only arise during transition from the first to the second case, i.e. when 1) a female join a lone male (or vice versa) and form a breeding pair, or 2) a lone male joins k females for $\forall k \in [2, n]$. Using the definition of monotonicity for each of two cases, we have:

$$\begin{cases} h \leq f(1) = a \\ h \cdot k \leq f(k), \forall k \in [2, n] \end{cases} \quad (8)$$

By transforming the second inequality to $h \leq \frac{f(k)}{k}$, and knowing that $f(k) \leq 1$ for $\forall k$ and $f(k) - f(k-1)$ is always less than 1 for $\forall k > 2$, we can conclude that $f(k)$ is increasing with k slower than k , implying that $\min(\frac{f(k)}{k}) = \frac{f(n)}{n} = \frac{1-t}{n}$. Hence, we can rewrite Equation (8) as the following:

$$\begin{cases} h \leq a \\ h \leq \frac{f(n)}{n} = \frac{1-t}{n} \end{cases} \quad (9)$$

The break of superadditivity could arise in two cases: 1) a female joins a lone male (or vice versa), and 2) a lone male joins k females for $\forall k \in [2, n]$ (3) s females join a coalition of a male and k females (or vice versa) for $\forall s, k : s+k \leq n$. Using superadditivity for each of three cases, we have:

$$\begin{cases} 2h \leq f(1) = a \\ hk + h = h(k+1) \leq f(k), \quad \forall k \in [2, n] \\ f(k) + s \cdot h \leq f(k+s), \quad \forall s, k \geq 1 : s+k \leq n \end{cases} \quad (10)$$

Since $\min(\frac{f(k)}{k}) = \frac{f(n)}{n} = \frac{1-t}{n}$ (see above), we can conclude that $\min(\frac{f(k)}{k+1}) = \frac{f(n)}{n+1} = \frac{1-t}{n+1}$. We rewrite the second inequality of Equation (10):

$$h \leq \frac{1-t}{n+1} \quad (11)$$

By transforming the third inequality of (10) to $h \leq \frac{f(k+s)-f(k)}{s}$ and

using Equation (1) such that $f(k) = a + (1-a) \cdot \frac{\sum_{i=1}^{k+s} y(i)-1}{d-1}$ we have

$$\begin{aligned} \frac{f(k+s) - f(k)}{s} &= \frac{1}{s} \cdot \frac{1-a}{d-1} \cdot \left(\sum_{i=1}^{k+s} y(i) - \sum_{i=1}^k y(i) \right) \\ &= \frac{1-a}{d-1} \cdot \frac{\sum_{i=k+1}^{k+s} y(i)}{s} \end{aligned} \quad (12)$$

where $\frac{\sum_{i=k+1}^{k+s} y(i)}{s}$ is the average of elements of series $y(k+1), \dots, y(k+s)$, and $y(i) = e^{-\gamma(i-1)}$ decreases with k and $k+s$ for $\gamma > 0$ and $k, s \geq 1$. Therefore the average is also decreasing with k and $k+s$ and equation $\frac{f(k+s)-f(k)}{s}$ reaches its minimum at $k+s = \max(k+s) = n$, $k = \max(k) = n-1$, $s = 1$. Therefore,

$$\begin{aligned} \min\left(\frac{f(k+s)-f(k)}{s}\right) &= f(n) - f(n-1) \\ &= \frac{1-a}{d-1} \cdot y(n) \\ &= \frac{1-a}{d-1} \cdot e^{-\gamma(n-1)} \\ &= \frac{t}{d-1} \end{aligned} \quad (13)$$

Using Equations (11) and (13), we rewrite superadditivity conditions (10) as:

$$\begin{cases} h \leq \frac{a}{2} \\ h \leq \frac{1-t}{n+1} \\ h \leq \frac{t}{d-1} \end{cases} \quad (14)$$

According to Theorem 3.1 the game has nonempty core if $h \leq \frac{1-t}{n+1}$. Since $\frac{1-t}{n+1} < \frac{1-t}{n}$ in the game with nonempty core $h \leq \frac{1-t}{n}$ as well. Therefore given the game with nonempty core, the second condition of monotonicity, Equation (9), always holds and only the first, $h \leq a$, remains active. The conditions for the existence of the core in Theorem 3.1 are the same as the second and the third conditions of superadditivity, Equation (14), so they are the subset of superadditivity conditions. Therefore superadditivity is sufficient for the existence of the core, and thus, for a game G with a nonempty core, $v(C)$ is superadditive if the remaining condition of Equation (14), $h \leq \frac{a}{2}$ holds true. \square

In coalitional game theory monotonicity and superadditiveness are neither necessary nor sufficient conditions for the existence of the core [11, 44]. Since in natural systems group formation processes are not always monotonic or superadditive, we analyze the stability of complex family in its more generic way as defined in Theorem 3.1, including for non-monotonic and non-superadditive cases.

3.1 The likelihood of complex family formation

We intentionally formulated the conditions of Theorem 3.1 in terms of the migration parameter, h , to provide a more in-depth analysis of complex family formation. From the biological standpoint, parameter h has an intriguing meaning and defines the benefits of migration. It justifies the impact of the external environment on the family: availability of vacant home ranges nearby and migrant endangerment, which in turn depend on local density and inner structure of a population.

Local density and structure are spatially and temporally variable on a population level, so does h – it is fixed for a given family for the given year but not for a population as a whole. In this context, \hat{h} (the maximum h value of nonempty core existence) implies the robustness of the grand coalition to the spatial variation and temporal changes of population characteristics. Therefore, we could expect that as \hat{h} increase the, so called, ‘likelihood’ of complex family coalitions increases because the potential range for h can vary

in a wider range without any threats to the stability of the grand coalition. Such favorable conditions increase the development of social mechanisms (leading to a complex family) as an evolutionary response [14, 24, 29].

To better illustrate the impact of this parameter, consider two instances A and B which occur in populations of identical size with the same distribution of migration parameter $h_A, h_B \sim \mathcal{N}(\bar{h}, \sigma)$ – a Gaussian distribution with the mean \bar{h} and variance σ^2 . Let A and B differ in parameters a, n, γ so that $\hat{h}_A > \hat{h}_B$. In this case the likelihood of a nonempty core for formation of the grand coalition in instance A is higher than that of B , that is, $P(h_A \leq \hat{h}_A) > P(h_B \leq \hat{h}_B)$. In other words, A would yield higher frequency of coalitions for which complex family formation is advantageous over smaller coalitions, which confirms the evolutionary fitness advantage of population genetics [14, 24, 29].

In the light of these results, a natural interesting question is how parameters a, n, γ influence \hat{h} and thus the likelihood of complex family formation.

3.2 Impact of pair utilization a on the core

Since a monogamous pair is an alternative to a complex family, the pair utilization a impacts the outcome in two ways: (1) the balance between a and migration parameter h largely determines the breeding pair’s stability with a total breakdown of the family (to lone migrants) as an alternative, and (2) the balance between a and marginal contribution of helper females determines relative advantages of a breeding pair comparing to a complex family. Below, we first show that a certain value of pair utilization, a , acts as a threshold between two states of the family formation game, splitting the space of the core into two.

LEMMA 3.3. *Given the number of females n and marginal contribution γ , there exists a threshold $a^*(n, \gamma)$ as follows*

$$a^*(n, \gamma) = 1 - \frac{d-1}{e^{-\gamma(n-1)} \cdot (n+d)} \quad (15)$$

such that if $a > a^*(n, \gamma)$ then $\hat{h} = \frac{t}{d-1}$ and if $a < a^*(n, \gamma)$ then $\hat{h} = \frac{1-t}{n+1}$.

PROOF. Let’s define combination of model parameters when the first condition of Theorem 3.1 defines \hat{h} :

$$\begin{aligned} \frac{t}{d-1} &< \frac{1-t}{n+1} \\ t &= (1-a) \cdot e^{-\gamma(n-1)} < \frac{d-1}{n+d} \\ a &> 1 - \frac{d-1}{e^{-\gamma(n-1)} \cdot (n+d)} = a^*(n, \gamma) \end{aligned}$$

Therefore, when $a > a^*(n, \gamma)$ the first element of Theorem 3.1 implies \hat{h} and when $a < a^*(n, \gamma)$ the second element of Theorem 3.1 implies \hat{h} . \square

In the following sections, when it is clear from the context we abuse the notation and write a^* instead of $a^*(n, \gamma)$.

Indeed, one could expect that as a increases (the proportion of the home range utilized by the breeding pair), the benefit to the other females decreases, which subsequently, lowers the likelihood of complex family formation. Moreover, as a increases the total value

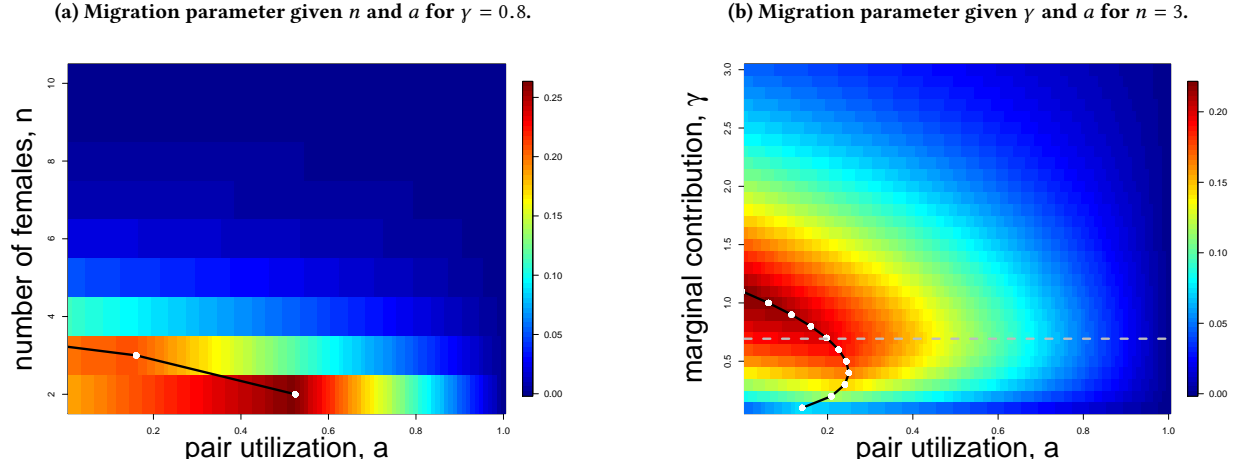


Figure 1: The space where core is nonempty with respect to the given parameters. Colors represent the value of \hat{h} - the maximum h value of nonempty core existence; the black line with white dots represent a^* - the value of a which corresponds to $\max \hat{h}$ for the given n, γ ; and the horizontal dashed line marks $\ln 2 \approx 0.7$.

of the grand coalition also increases. For $a < a^*$ the second (positive) effect outweighs the first (negative) and the likelihood of the complex family formation rises with a , which is a counterintuitive property of the game that we show next.

THEOREM 3.4. *Given n and γ , the highest likelihood of the complex family formation (the highest \hat{h}) is at $a = a^*$, or if $a^* < 0$ it is at $a = 0$.*

PROOF. According to Lemma 3.3 for $a > a^*$ the first element of Theorem 3.1 defines \hat{h} . Taking the partial derivative of the first element in Theorem 3.1 with respect to a results in negative expression which does not depend on a , implying a linear decrease with a , that is,

$$\begin{aligned} \frac{\partial \left(\frac{1-a}{d-1} \cdot e^{-\gamma(n-1)} \right)}{\partial a} &= -\frac{e^{-\gamma(n-1)}}{d-1} \\ \left[d-1 = \frac{e^{-\gamma}}{(1-e^{-\gamma})} \right] \\ &= -(1-e^{-\gamma}) \cdot e^{-\gamma(n-2)} < 0 \end{aligned}$$

For $a < a^*$, the partial derivative of the second element of Theorem 3.1 results in a positive expression, independent of a , that is,

$$\frac{\partial \left(\frac{1-(1-a)e^{-\gamma(n-1)}}{n+1} \right)}{\partial a} = \frac{e^{-\gamma(n-1)}}{n+1} > 0$$

As \hat{h} rises for $a < a^*$ and decreases for $a > a^*$, its value reaches the maximum at $a = a^*$. When $a^* < 0$ the lowest possible a ($a = 0$) results in the maximum value of \hat{h} . \square

4 EMPIRICAL RESULTS

We complement our analytical findings with empirical evaluation of the family formation game on a real-world case study: the family formation in the Arctic fox subspecies of Mednyi Island. While the Arctic fox population in the mainland (e.g. Russia, Canada, big

islands: Greenland and Iceland) are almost entirely monogamous, complex polygynous families are common in several Pacific islands such as Mednyi Island [18–20, 25].

4.1 The Core

We ran simulations to investigate the space of the core existence (combinations of parameters corresponding to nonempty core) by solving directly the linear program (2). The default parameters are set to $\gamma = 0.8$ and $n = 3$ based on evidence and reports from Mednyi Island fox family structures: complex families in 60% of cases are composed of 2 females and in 30% of 3 females; so families with more than 3 females are relatively rare [20]. If such a ratio (families with 3 females appear two times less frequently than with 2 females) correlates with marginal contribution of the second and third female then the diminishing factor $\gamma = 0.8$ may adequately describe such situation as $\frac{f(k)-f(k-1)}{f(k-1)-f(k-2)} = e^{-0.8} \approx 0.45 \approx 0.5$ for $\forall k > 2$.

In our simulations, we varied pair utilization a and one of the parameters γ or n , while fixing the other at the default level, and further computed a^* . The variation intervals were $n \in [2, 10]$; $\gamma \in [0.1, 2]$; $a \in [0, 1]$ and include all values that has a biological meaning. Given n, γ, a , we found all values of h for which nonempty core exists and find the maximum value, corresponding to \hat{h} , that is, the likelihood of a complex family formation in a population (see section 3.1)

Fig. 1 illustrates the space of the core in 3 dimensions, where the third dimension, color, represents \hat{h} such that larger \hat{h} (color closer to red) is interpreted as a higher likelihood of complex family formation of size n . The space of the core is split into two zones by plane $a^*(n, \gamma)$ (in Fig. 1, a plane turns into a line in a 3D slice).

The lines in Figure 1, a^* , splits the first (left of the line) and the second (right) conditions of Theorem 3.1. When $a = a^*$, the likelihood of complex family formation is maximized for any given

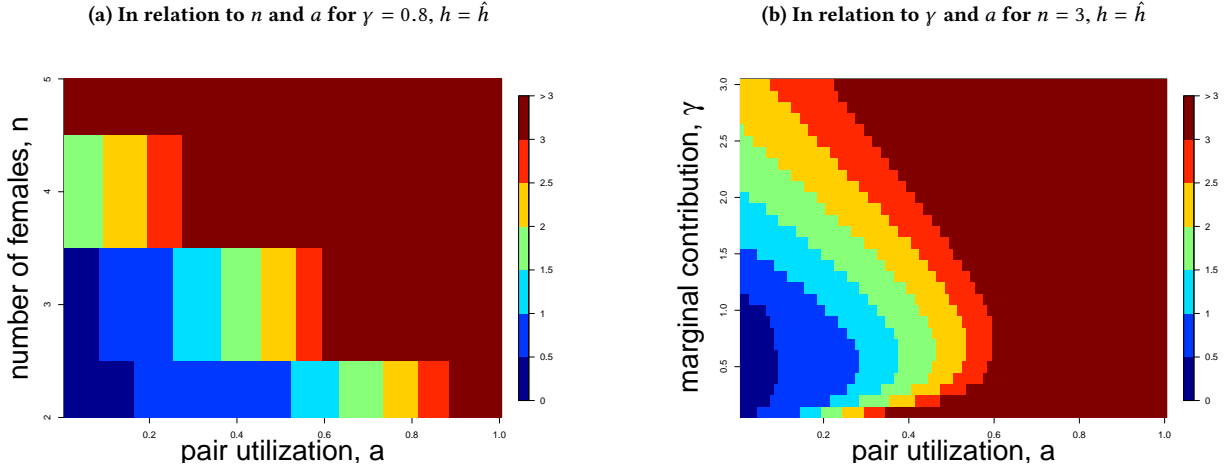


Figure 2: The male to female ratio of the Shapley value with respect to the given parameters. Colors represent the male to female Shapley ratio. We used $h = \hat{h}$ – the maximum value of h for the nonempty core.

n, γ (see Theorem 3.4). Fig. 1 shows that a^* rapidly decreases with n and γ (after a slight growth) and reaches negative values; hence, the space of the core existence with active second condition of Theorem 3.1 is relatively small.

The maximum value of \hat{h} in simulations is $0.20 - 0.25$: $\frac{1}{4}$ to $\frac{1}{5}$ of the home range capacity. This interval sets a theoretical threshold for the migration parameter, h , above which a complex family in a model population could not be stable. Fig. 1 also reveals the same type of threshold for the pair utilization parameter, a , at $0.8 - 0.9$ (depending on n and γ): if a breeding pair utilizes more than 80%-90% of the home range capacity, a complex family formation of any size is highly improbable (turquoise and blue colors: $h < 0.05$).

Fig. 1a clearly shows the optimum size for the stability of complex family which is the number of female helpers $n \in \{2, 3\}$. Interestingly, for $a < 0.2$ the stability of families with $n = 2$ and 3 is nearly identical (actually a family with 3 females is a bit more stable); hence both structures could coexist in a population. For $a < 0.2$, the existence of families with $n = 4$ is also realistic though with much lower likelihood. As a increases, especially for $a > 0.5$, complex families with $n = 2$ become distinctively more probable than others. Fig. 1a shows that the theoretical maximum size of a complex family lies at $n \in \{5, 6\}$, above which likelihood of family formation approaches zero (dark blue color: $h < 0.01$, margin of error in our simulation).

Fig. 1b demonstrates that the space of the nonempty core consists of isopleths (sections of the same color, \hat{h} value) in a plane γ, a which looks parabolic with extrema at $\gamma = \ln\left(\frac{n-1}{n-2}\right) = \ln 2 \approx 0.7$. The line a^* (black with white dots) have a similar shape but with the extrema at $\gamma \approx 0.4, a = 0.25$. The relative high likelihood of complex family formation with 3 females ($h > 0.1$) corresponds to the wide interval of marginal contribution diminishing factor, $\gamma = 0.2 - 2.0$. Indeed in range $0.2 - 1.2$ (neighborhood of $\gamma = \ln 2$) the likelihood is very robust to changes in γ . Such insensitivity yields fruitful properties to our framework as in reality γ is quite challenging to

estimate accurately. For $n = 2$, the likelihood of complex family formation monotonically increases with γ ; for $\gamma > 0.7, a < 0.8$ the grand coalition with 2 females becomes fairly stable ($\hat{h} > 0.2$) and insensitive to γ .

4.2 Agents' Power: Shapley Value

There are several plausible ways to measure the importance/power of agents in a coalition [36]. We adopt the most common solution concept of Shapley Value, that aims to measure the importance of each agent to the grand coalition by computing the weighted average of each agent's marginal contribution. The Shapley value for a breeding male $m, \phi(m)$, in the game G is calculated as follows:

$$\phi(m) = \frac{1}{(n+1)!} \left(\sum_{k=1}^n C_n^k k! (n-k)! (f(k) - hk) + n! h \right) \quad (16)$$

where k is the number of females in the coalition before m joins in.

In Equation (16) the first term in the main parentheses (the sum) represents cases of a male joining the grand coalition when it already has at least one female – its marginal contribution $f(k) - hk$ multiplied by (1) the number of different ways we can select these k females, C_n^k (2) the number of permutations of females before, $k!$, and (3) after, $(n-k)!$, the male joins the coalition. The second term of Equation (16) represents cases when a male is the first member of the coalition - a male brings h multiplied by $n!$ sets of females that join the coalition after him. Shapley value of a female is computed using the same principles (formula not shown). Marginal contribution of j th female depending on the order of joining could be 1) h if j th female joins a coalition consisting only from females 2) $f(k+1) - f(k)$ if it joins coalition of a male and $k \geq 1$ females and 3) $a - h$ if it joins the lone male.

Figure 2 illustrates the male to female importance ratio. We used the same default parameters as for the space of the core (see Section 4.1) with migration parameter $h = \hat{h}$ when h is fixed (Figures 2a

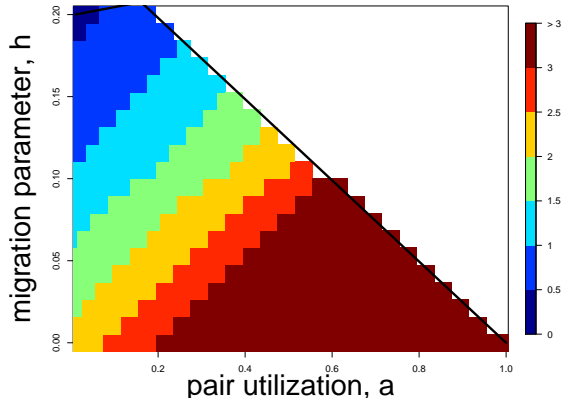


Figure 3: The male to female ratio of the Shapley value with respect to h and a for $n = 3, \gamma = 0.8$. The black line represents $h = \hat{h}$ and the white space illustrates the space where the core is empty.

and 2b). Predictably, the male’s power (or importance) increases with the number of females n and pair utilization a . Interestingly, when $\gamma = 0.8$, a female’s contribution could exceed that of a male agent (ratio less than 1) for $n = 2$ and $n = 3$ in case of $a < 0.50$ and < 0.25 respectively (blue colors on Figure 2a). For $n = 3$, the relative contribution of a female is the highest for the interval $\gamma = 0.5 - 1.0$ (Fig. 2b).

A female’s leverage to the grand coalition increases with migration parameter h as its threat to leave the coalition becomes stronger; though for $n = 3$ and $\gamma = 0.8$ a female’s Shapley value exceeds the male’s value in a narrow range: $a < 0.25$ and $0.12 < h < 0.2$ (Figure 3). Consider the example of $n = 3, \gamma = 0.8, a = 0.25$, and $h = 0.18$ (this combination of parameters represent a nonempty core), meaning a family of one male and 3 females in an environment where migration benefits are just marginally below the payoff for a potential breeding pair. In this case the male to female Shapley value ratio gets close 1: the importance of the male agent is indistinguishable from any of the 3 females. Nonetheless, in a similar case that only differs in a ($a > 0.6$), this ratio exceeds 3 for all $h > 0.01$ (margin of error in our simulation): the male’s importance is significantly higher than those of all 3 female agents combined (Figure 3).

4.3 Ecological Ramifications

Our empirical results confirms the reports about family structures of Medniy Arctic fox subspecies in case of the pair utilization parameter at $0.5 < a < 0.7$ (Fig. 1a): complex families with 2 females are more frequent than with 3 females, and groups with more than 4 females are nearly absent. Our results suggest that in the mainland where complex families are absent, a should be very high (> 0.8), which corresponds to lower intruder pressure compared to that of Mednyi Island. This is likely the case because the population density of foxes in the mainland is tens of times smaller than that of Mednyi Island [20].

More importantly, our model sheds light on the historical disease outbreak in this particular subspecies of foxes that wiped out nearly 90% of the population between 1970–1980 [18]. The high density population in that period implies a very low pair utilization ($a < 0.2$ as we believe), suggesting that breeding pairs were not able to defend their territories without helper females. According to our empirical analysis, this low pair utilization a gives rise to new family compositions with the equal proportion of families with 2 and 3 female helpers. Interestingly, according to our estimates such low a shifts the balance of power (measured as Shapley value) towards females (Figure 2 for $a < 0.25$). Our observations of modern population of Mednyi foxes do not reveal strong dominance of a male in the family despite its larger body. We hypothesize that observed phenomenon could be an artifact of the recent past.

5 CONCLUDING REMARKS

Inspired by a subspecies of foxes, we studied the coalition formation games with complex multiagent structures. We analyzed the formation of such complex coalitions through the lenses of multiagent systems and game theory, and provided theoretical and empirical justifications about the conditions under which stable complex families arise. Our theoretical analysis enables researchers to study the discovered parameters in order to predict future ramifications of resource scarcity on animal behavior and estimate the sensitivity of populations to environmental and ecological changes.

Our model can be used in other multiagent settings for team formation that deal with precious and scarce resources [2, 12]. For instance, our coalitional model can be used to effectively allocate human resources in search and rescue processes where heterogeneous agents must form teams with certain combined skills to be dispatched [35]. For instance, in most rescue operations, teams have minimum requirements for their size, and sometimes skills, which is analogous to a breeding pair, with expected diminishing values added to the coalition with each additional member. In the future, we plan to further generalize our model to other application domains and extend it to environments with uncertain parameters.

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