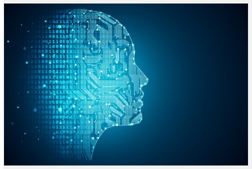

Data Science for Researchers and Scholars

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Random Variables

- A variable x is a random variable if the value that it assumes, corresponding to the outcome of an experiment is a chance or random event.
- Random variables can be discrete or continuous

Examples

- x = SAT score for a randomly selected student
- x = number of people who click on your website on a randomly chosen of the year 2023
- x = outcome of a die toss

Probability Distributions of Discrete Random Variables

- The probability distribution for a discrete random variable *x* is a graph, table or formula that gives the probability $p(x)$ associated with each value of x
- Note that
  - $\forall x \; 0 \leq p(x) \leq 1$
  - $\sum_x p(x) = 1$

Example

Toss a fair coin three times and define x = number of heads.

		x
HHH	1/8	3
HHT	1/8	2
HTH	1/8	2
THH	1/8	2
HTT	1/8	1
THT	1/8	1
TTH	1/8	1
TTT	1/8	0

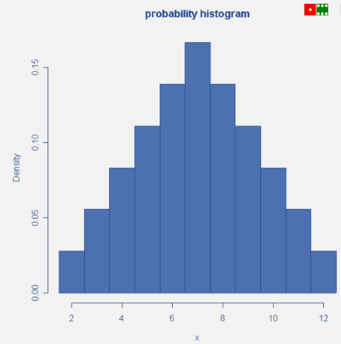
$P(x = 0) = 1/8$
$P(x = 1) = 3/8$
$P(x = 2) = 3/8$
$P(x = 3) = 1/8$

x	$p(x)$
0	1/8
1	3/8
2	3/8
3	1/8

Probability distribution of x

Example

Toss two dice and define
x = sum of two dice.



x	$p(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

Probability Distributions

- Probability distributions can be used to describe the population, just as we described samples using statistics
- Shape: Symmetric, skewed, mound-shaped…
  - Outliers: unusual or unlikely measurements
  - Center and spread: mean and standard deviation. A population mean is called μ and a population standard deviation is called σ.
- Let x be a discrete random variable with probability distribution $p(x)$. Then the mean, variance and standard deviation of x are given as

$$\text{Mean}: \mu = \sum x p(x)$$
$$\text{Variance}: \sigma^2 = \sum (x-\mu)^2 p(x)$$
$$\text{Standard deviation}: \sigma = \sqrt{\sigma^2}$$

Example

Toss a fair coin 3 times and record x, the number of heads.

x	$p(x)$	$xp(x)$	$(x-\mu)^2 p(x)$
0	1/8	0	$(-1.5)^2(1/8)$
1	3/8	3/8	$(-0.5)^2(3/8)$
2	3/8	6/8	$(0.5)^2(3/8)$
3	1/8	3/8	$(1.5)^2(1/8)$

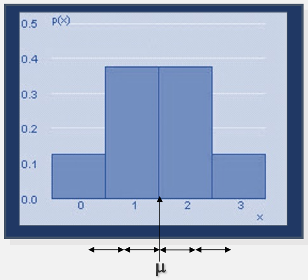
$$\mu = \sum xp(x) = \frac{12}{8} = 1.5$$

$$\sigma^2 = \sum (x-\mu)^2 p(x)$$

$$\sigma^2 = .28125 + .09375 + .09375 + .28125 = .75$$
$$\sigma = \sqrt{.75} = .688$$

Example

The probability distribution for *x* the number of heads in tossing 3 fair coins.



- Shape? – Symmetric
- Outliers? - None
- Center? $\mu = 1.5$
- Spread? $\sigma = .688$

Random Variable and Distribution

- A random variable X is an outcome of a random experiment
- The distribution of a random variable is a table, graph or a formula that gives the probability $P(X)$ associated with each possible value of X
  - In the case of discrete random variables, we write

$$P(X = x) = p_\theta(x)$$

  to mean probability that the random variable X takes the value x is $p_\theta(x)$, a function of x, parameterized by θ.
  - $\forall x \; 0 \leq p_\theta(x) \leq 1$
  - $\sum_x p_\theta(x) = 1$

Bernoulli distribution

- Bernoulli distribution is a discrete probability distribution
- It escribes the probability of achieving a "success" or "failure" from a random experiment (called Bernoulli trial) with only two possible outcomes (success or failure).
- Example: outcome of coin toss with two outcomes – heads (success denoted by 1) or tails (denoted by 0)

$$P(X = x) = \theta^x (1-\theta)^{1-x}$$

  - When $x = 1$, we have
    $P(X = 1) = \theta^1 (1-\theta)^{1-1} = \theta$
  - When $x = 0$, we have
    $P(X = 0) = \theta^0 (1-\theta)^{1-0} = 1 - \theta$
  - Note that $\forall \theta$ such that $0 \leq \theta \leq 1$,
    - $\forall x \; 0 \leq p_\theta(x) \leq 1$ and
    - $\sum_x p_\theta(x) = \theta + 1 - \theta = 1$

Exercise: Mean and variance of Bernoulli distribution

Exercise: Mean and variance of Bernoulli distribution

$$P(X=x)=\theta^x(1-\theta)^{1-x}$$

- Mean = expectation of x

$$\mu=\sum_x xP(X=x)=1(\theta)+0(1-\theta)=\theta$$

- Variance = expectation of the square of the difference between x and the mean of x

$$\sigma^2=\sum_x (x-\mu)^2 P(X=x)$$

$$\sigma^2 \;=(1-\mu)^2\theta+(0-\mu)^2(1-\theta)$$

$$\sigma^2 \;=\theta-\mu^2\theta-\mu^2+\mu^2\theta=\theta-\theta^2=\theta(1-\theta)$$

Categorical distribution generalizes Bernoulli distribution

- Instead of 2 outcomes, now we have k discrete outcomes $1, 2, \cdots k$ that occur with probabilities $p_1, p_2, \cdots p_k$
- Example: outcome of k-sided die toss

$$P(X = x) = p_1^{\mathrm{I}(x=1)}\, p_2^{\mathrm{I}(x=2)} \cdots p_k^{\mathrm{I}(x=k)}$$

where

$\mathrm{I}(x = v)$=1 iff $x = v$ and $\mathrm{I}(x = v)$=0 otherwise

Note that

$P(X = 1) = p_1$, $P(X = 2) = p_2$,$\cdots$ $P(X = k) = p_k$ as desired

We further require that $\forall k\ 0 \leq p_k \leq 1$ and $\sum_{v=1}^{k} p_v = 1$

Categorical distribution

- A convenient way to represent the outcome of a categorical random experiment is one hot encoding, a k-element vector with a 1 in the position corresponding to the observed outcome and 0s everywhere else.
  - Outcome $X = 1 = x_1$is encoded as $\mathbf{v}_1 \; = [1,0,0, \cdots 0]$
  - Outcome $X = 2 = x_2$ is encoded as $\mathbf{v}_2 \; = [0,1,0, \cdots 0]$ …
  - Outcome $X = k = x_k$ is denoted by $\mathbf{v}_k \; = [0,0,0, \cdots k]$
- Now
  - $\mathbf{v}_1$ occurs with probability p_1
  - $\mathbf{v}_2$ occurs with probability p_2 …
  - $\mathbf{v}_\mathrm{k}$ occurs with probability p_k
- The outcomes of the categorical random variable X have a 1-1 correspondence with one-hot vector valued random variable $\mathbf{V}$
- One hot encoding offers many conveniences
  - As an exercise, compute the mean of the categorical distribution with
    - Scalar discrete representation of the outcomes
    - One hot encoding of the outcomes

Categorical distribution

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  - $\mathbf{v}_1$ occurs with probability p_1
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  - $\mathbf{v}_\mathrm{k}$ occurs with probability p_k
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One hot encoding of categorical outcomes

x → One Hot Encoder → $\mathbf{v}$

Example: 6-sided die

3 → One Hot Encoder → $\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$

- The correspondence between x and $\mathbf{v}$ is a bijection
- x and $\mathbf{v}$ contain the identical information
  - Outcome of the categorical random experiment
  - Example, outcome of tossing a 6-sided die

Mean and variance of Categorical distribution

Discrete scalar representation of outcomes

$$P(X = x) = p_1^{\mathrm{I}(x=1)}\, p_2^{\mathrm{I}(x=2)} \cdots\, p_k^{\mathrm{I}(x=k)}$$

- Mean = expectation of X
- $\mu = \sum_i \; x_i\, P(X = x_i) = 1p_1 + 2\, p_2 + \cdots + kp_k$

One hot vector representation of outcomes

$\forall i \in \{1, \cdots k\}, P(\mathbf{V} = \mathrm{v}_i) = p_i$

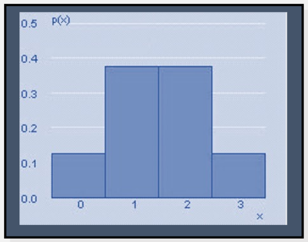
- Mean = expectation of $\mathbf{V}$
- $\boldsymbol{\mu} = \sum_i \; \mathrm{v}_i P(\mathbf{V} = \mathrm{v}_i) = \sum_i \; \mathrm{v}_i p_i = [p_1, p_2, \cdots p_k]$

- One hot encoding is elegant and offers many conveniences
- We will use it often in machine learning

The Binomial Random Variable

- Binomial random variable generalizes the Bernoulli variable
- Bernoulli – Toss a coin once and record the outcome
- Toss a coin *n* times and record x = number of heads

Binomial distribution of the number of heads in 3 tosses of a fair coin



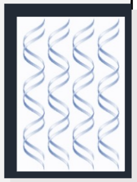
x	$p(x)$
0	1/8
1	3/8
2	3/8
3	1/8

Bernoulli versus Binomial

- The Bernoulli distribution represents the success or failure of a single Bernoulli trial.
- The Binomial Distribution represents the number of successes and failures in n independent Bernoulli trials for some given value of n.

The Binomial Random Variable

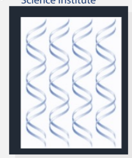
- Many situations in real life resemble the coin toss, but the coin is not necessarily fair, so that P(H) $\neq$ 1/2.
- Example: A geneticist samples 10 people and counts the number who have APOE-e4 a gene linked to Alzheimer's disease.
- Coin: Person
- Head: Has one or more copies of APOE-e4 gene
- Tail: Has no copy of APOE-e4 gene
- Number of coin tosses: $n = 10$
- P(Has Alzheimer's gene) = P(H) = fraction of the population that has at least 1 copy of the APOE-e4 gene ≈ 0.2 to 0.3



The Binomial Experiment

- The experiment consists of n identical trials.
- Each trial results in one of two outcomes, success (S) or failure (F).
  - The probability of success (or failure) on a single trial is p and the probability of failure is $q = 1 - p$.
- The probabilities p (and hence q) remain constant from trial to trial.
- The trials are independent.
- We are interested in x, the number of successes in n trials.

Binomial or Not?



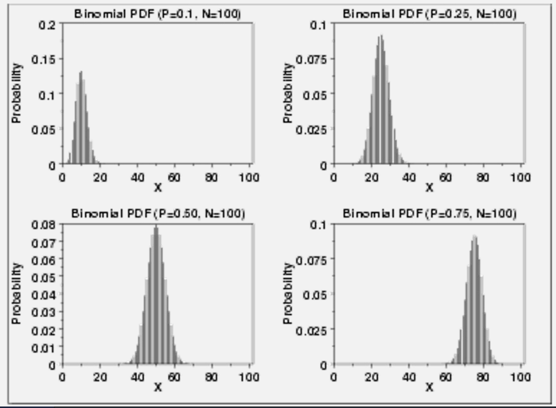
- Binomial distribution requires that the trials be independent
- Independence is often violated in real life applications

- Select two people from the U.S. population, and suppose that 20% of the population has the APOE-e4 Alzheimer's gene.
  - For the first person, $p = \text{P(gene)} = 0.20$
  - For the second person, $p \approx \text{P(gene)} = .20$, even though one person has been removed from the population.

Binomial or Not?

- 1 in 10 PCs are defective.
- We have 20 PCs in the lab
- We randomly select 3 for testing.
- Is this a binomial experiment?
  - The experiment consists of $n = 3$ identical trials
  - Each trial results in one of two outcomes
  - The probability of success (finding the defective PC) is 0.1 and it remains constant across trials
  - But there is a catch.
    - The trials are not independent.
    - P(success on the 2nd trial | success on the 1st trial) = $1/19$, not $2/20$
- Rule of thumb: if the sample size n is large relative to the population size N, say $n/N \geq .05$, the trials are likely not independent and the experiment not likely binomial.

Plots of Binomial Distribution



The Binomial Probability Distribution

- For a binomial experiment with n trials and probability p of success on any single trial, the probability of k successes in n trials is

Number of ways to choose k out of n items

$$P(x=k) = C_k^n p^k q^{n-k} = \frac{n!}{k!(n-k)!} p^k q^{n-k} \text{ for } k = 0,1,2,\ldots n.$$

Recall $C_k^n = \dfrac{n!}{k!(n-k)!}$

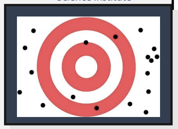
with $n! = n(n-1)(n-2)\ldots(2)1$ and $0! \equiv 1$.

Mean and Standard Deviation: Binomial Distribution

Exercise: For a binomial experiment with n trials and probability p of success on a given trial, show that

- Mean $\mu = np$
- Variance $\sigma^2 = npq$
- Standard deviation $\sigma = \sqrt{npq}$

Example



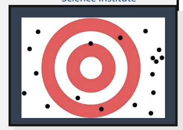
- Ukrainian missiles hit a target 80% of the time.
- The Ukrainian forces fire five missiles at a target.
- What is the probability that exactly 3 missiles hit the target?

$n = 5$ $\quad$ *success* = hit $\quad$ $p = .8$ $\quad$ $x =$ # of hits

$$P(x=3) = C_3^n p^3 q^{n-3} = \frac{5!}{3!2!}(.8)^3(.2)^{5-3}$$

$$= 10(.8)^3(.2)^2 = .2048$$

Example (Contd)



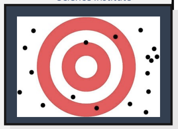
What is the probability that more than 3 missiles hit the target?

$$P(x > 3) = C_4^5 p^4 q^{5-4} + C_5^5 p^5 q^{5-5}$$

$$= \frac{5!}{4!1!}(.8)^4(.2)^1 + \frac{5!}{5!0!}(.8)^5(.2)^0$$

$$= 5(.8)^4(.2) + (.8)^5 = .7373$$

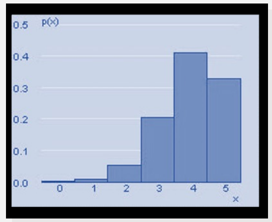
Exercise



- What is the probability that no missiles hit the target?
- What is the probability that fewer than 3 missiles hit the target?
- What is the probability that fewer than 4 but more than 1 missiles hit the target?

Exercise

- Plot the probability distribution for x = number of hits.
- What are the mean and standard deviation for x?



Mean $\mu = np = 5(0.8) = 4$
Standard deviation $\sigma = \sqrt{npq} = 0.89$

The Poisson Random Variable

- The Poisson random variable is often used to model the number of occurrences of a specified event in a given unit of time or space.

- Examples:
  - The number of calls received by a switchboard during 9am to 5pm.
  - The number of times a printer jams in a day
  - The number of traffic accidents at the intersection of Atherton Street and College Avenue in State College on a football weekend.

The Poisson Probability Distribution

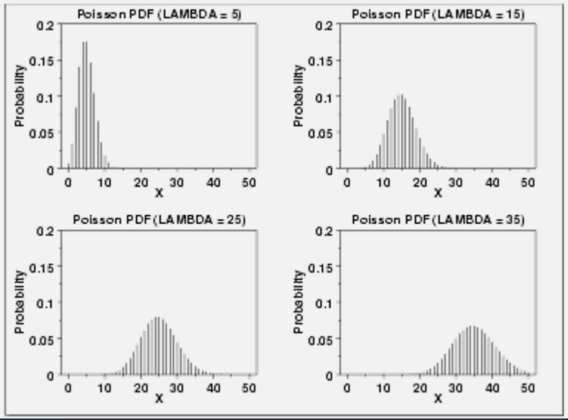
- Let x a Poisson random variable.
- The probability that $X = k$ (k occurrences of the event of interest) $for\ k = 0,1,2,\cdots$ is given by:

  $P(X = k) = \frac{\mu^k e^{-\mu}}{k!}$ for $k \geq 0$

  $P(X = k) = 0$ otherwise

  - Where μ is the mean of the distribution and standard deviation is $\sqrt{\mu}$ and $e \approx 2.718281828459$ is the base of the natural logarithm
  - We get Poisson by fixing the mean of the Binomial distribution, and letting the number of trials approach ∞

Plots of Poisson Distribution



Exercise

- The average number of traffic accidents on a certain intersection in New York city is two per week.
- What is the probability of exactly one accident during a one-week period?

$$P(x=1)=\frac{\mu^k e^{-\mu}}{k!} \quad =\frac{2^1 e^{-2}}{1!}=2e^{-2}=.2707$$

Exercise

- What is the probability that 8 or more accidents happen during a 1-week period?
- What is the probability that no accidents happen during a 1-week period?

The Hypergeometric Probability Distribution

- A bowl contains M red M&M candies and $N - M$ blue M&M® candies.
- Select n candies from the bowl (without replacement) and record x the number of successes - red M&Ms selected.
- Why can't we use the Binomial distribution? – trials are not independent
- Hypergeometric distribution is given by $P(X = k) = \frac{C_k^M C_{n-k}^{N-M}}{C_n^N}$
  - Where N is the population size
  - M is the maximum number of possible successes
  - n is the number of trials
  - k is the number of successes

Mean and Variance of Hypergeometric distribution

$$\text{Mean}: \mu = n\left(\frac{M}{N}\right)$$

$$\text{Variance}: \sigma^2 = n\left(\frac{M}{N}\right)\left(\frac{N-M}{N}\right)\left(\frac{N-n}{N-1}\right)$$

Exercise

- A package of 8 AA batteries contains 2 batteries that are defective.
- A student randomly selects 4 batteries and replaces the batteries in his calculator.
- What is the probability that all four batteries work?

Success = working battery

$$N = 8$$
$$M = 6$$
$$n = 4$$
$$k = 4$$

$$P(x=4) = \frac{C_4^6 C_0^2}{C_4^8}$$

$$= \frac{6(5)/2(1)}{8(7)(6)(5)/4(3)(2)(1)} = \frac{15}{70}$$

Exercise

- What are the mean and variance for the number of batteries that work?

$$\mu = n\left(\frac{M}{N}\right) = 4\left(\frac{6}{8}\right) = 3$$

$$\sigma^2 = n\left(\frac{M}{N}\right)\left(\frac{N-M}{N}\right)\left(\frac{N-n}{N-1}\right)$$
$$= 4\left(\frac{6}{8}\right)\left(\frac{2}{8}\right)\left(\frac{4}{7}\right) = .4286$$

Continuous Random Variables

- A random variable is continuous if it can assume the infinitely many values corresponding to points on a line interval.
- Examples
  - Height, weight
  - Scores on a test
  - Measurement error