DS 310 Machine Learning

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Function approximation (Regression)

- Function approximation is like classification except the labels are real valued

Example applications:

Predicting

- Stock value
- Income
- Power consumption

K nearest neighbor Function Approximator

Learning Phase

For each training example $(X_i, f(X_i))$, store the example in memory

Approximation phase

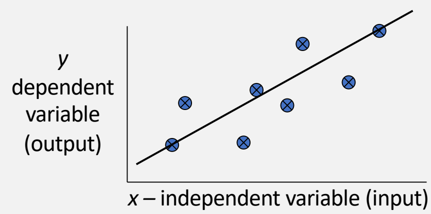
Given a query instance X_q, identify the k nearest neighbors $X_1 \ldots X_k$ of X_q

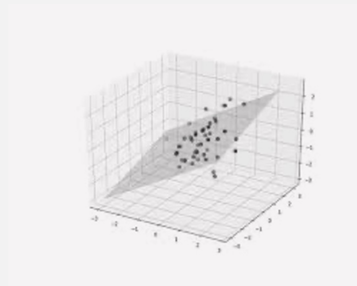
$$g(X_q) \leftarrow \frac{\sum_{l=1}^{K} f(X_i)}{K}$$

Value of a function (e.g., price of a product) at a query point is simply the average or inverse distance weighted average of the value of the function at the k nearest neighbors of the query point

Regression

- For classification the output is nominal
- In regression the output is continuous
- Linear regression is perhaps the simplest approach
  - Fit data with the best hyper-plane (line when the function is defined with respect to a single variable) which "goes through" the points





Simple Linear Regression

- In the simplest case, we have one (input), independent variable *x*, and one (output) dependent variable *y*
  - Multiple linear regression assumes an input vector **x**
  - Multivariate linear regression assumes an output vector **y**
- We will "fit" the points with a linear hyper-plane (line in the simplest case)
- Which line should we use?
  - Choose an objective function
  - For simple linear regression we choose sum squared error (SSE)
    - $\Sigma (d_i - y_i)^2 = \Sigma (e_i)^2$
  - Thus, find the line which minimizes the sum of the squared residues (e.g. least squares)

Hyperplane for multiple linear regression

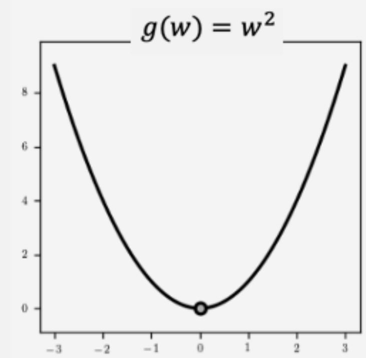
Digression: Minimizing functions

Minima of a function

- In many applications, machine learning included, we are often interested in minimizing a function of many variables.
- For a function $g(\mathbf{w})$ of N variables $w_1 \cdots w_N$, this problem is formally phrased as

$$\underset{\mathbf{w}}{\text{minimize}}\; g(\mathbf{w})$$

- That is, examine the value of $g(\mathbf{w})$ over all possible values of $\mathbf{w}$ in the domain of $g(\mathbf{w})$ and pick one or more where the value of $g(\mathbf{w})$ is minimum (over the range of $g(\mathbf{w})$.



- What is the smallest value of $g(w) = w^2$?
- What is the value of w at which this occurs?

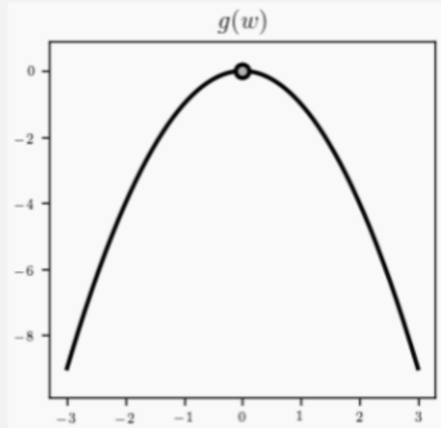
- Obviously, the minimum is smaller than any other value of the function.
- Specifically the smallest value - the global minimum of this function - seemingly occurs close to $w^{\star} = 0$
- Formally a point $w^{\star}$ gives the smallest point on the function if

$$g(w^{\star}) \leq g(w) \quad \text{for all } w.$$

- This is called the *zero-order definition of a global minimum* of a function.

- Suppose we multiply the quadratic function in the previous example by -1.
- The function flips upside down - now its global minima lie at $w^\star = \pm\infty$
- Now the point $w^\star = 0$ that used to be a global minimum is now global maximum - i.e., where the value of the function is the largest, i.e.,

$$g(w^\star) \geq g(w) \text{ for all } w.$$

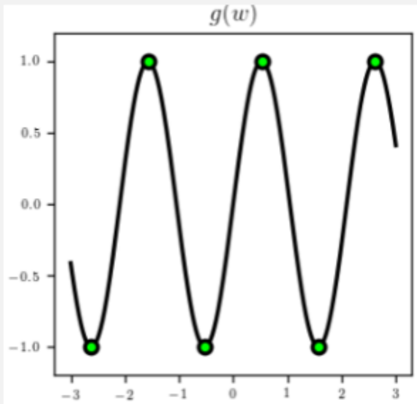


- These concepts of *minima* and *maxima* of a function are always related to each via multiplication by ***-1***.
- That is, any point that is a minima of a function ***g*** is a maxima of the function ***-g***, and vice-versa.

$$\underset{\mathbf{w}}{\text{maximize}} \;\; g\left(\mathbf{w}\right) = -\underset{\mathbf{w}}{\text{minimize}} \;\; g\left(\mathbf{w}\right).$$

Example

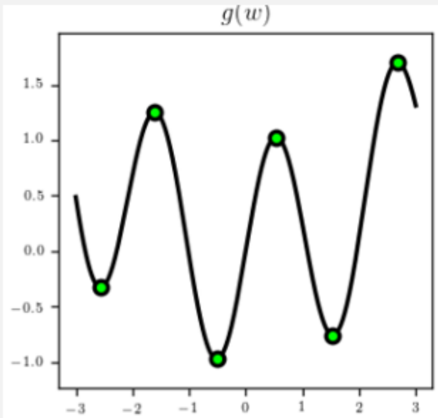
- Let us look at the sinusoid function $g(w) = \sin(2w)$
- Over the range we have plotted the function - that there are three global minima and three global maxima (marked by green dots).



- Technically speaking, this function has an infinite number of global maxima and minima
- Why?

Example: Minima and maxima of the sum of a sinusoid and a quadratic

- Let's look at a weighted sum of the previous two examples, the function $g(w) = \sin(3w) + 0.1w^2$ over a region of its input space.



- We have a global minimum around $w^\star = -0.5$ and a global maximum around $w^\star = 2.7$
- The point around $w^\star = 0.8$ is a local maximum.
- The point around $w^\star = 1.5$ is a local minimum
- Can you identify other local minima/maxima?

The zero order condition for optimality

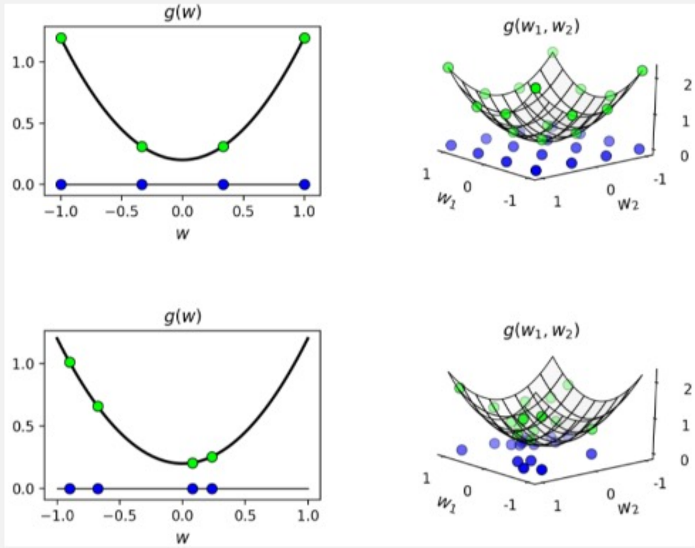
A point $\mathbf{w}^\star$ is

- a global minimum of $g(\mathbf{w})$ if and only if $g(\mathbf{w}^\star) \leq g(\mathbf{w})$ for all $\mathbf{w}$
- a global maximum of $g(\mathbf{w})$ if and only if $g(\mathbf{w}^\star) \geq g(\mathbf{w})$ for all $\mathbf{w}$
- a local minimum of $g(\mathbf{w})$ if and only if $g(\mathbf{w}^\star) \leq g(\mathbf{w})$ for all $\mathbf{w}$ near $\mathbf{w}^\star$
- a local maximum of $g(\mathbf{w})$ if and only if $g(\mathbf{w}^\star) \geq g(\mathbf{w})$ for all $\mathbf{w}$ near $\mathbf{w}^\star$

- Why zero order? Because it depends only on the function $g(\mathbf{w})$ and nothing else.
- Higher order definitions refer to the values of first, second … order derivatives of $g(\mathbf{w})$

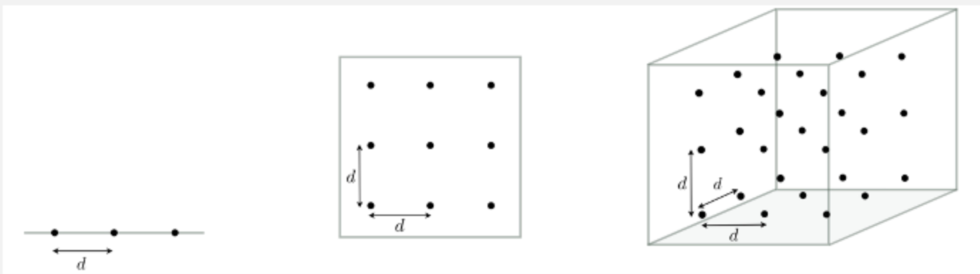
Finding global maxima / minima: A naïve approach

- Evaluate the function at a large number of points
- Among them, choose the input at which the function is the lowest as the approximate global minimum
- How can we choose the points at which to evaluate the function?
  - Uniformly
  - Randomly
- While this naïve approach works for functions of few variables, it fails for functions of more than 2 or 3 variables
- Why? The number of points at which the function needs to be evaluated grows exponentially with the number of variables


$g(w)$
w
$g(w_1, w_2)$
w_1
w_2
$g(w)$
w
$g(w_1, w_2)$
w_1
w_2

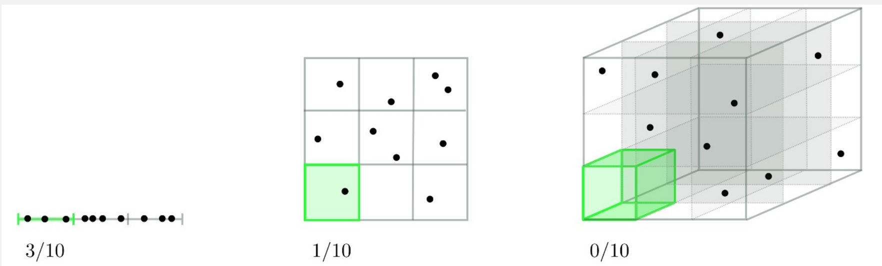
Curse of dimensionality

- While the naïve approach works for functions of few variables, it fails for functions of more than 2 or 3 variables
- Why? If sampled uniformly, the number of points at which the function needs to be evaluated grows exponentially with the number of variables



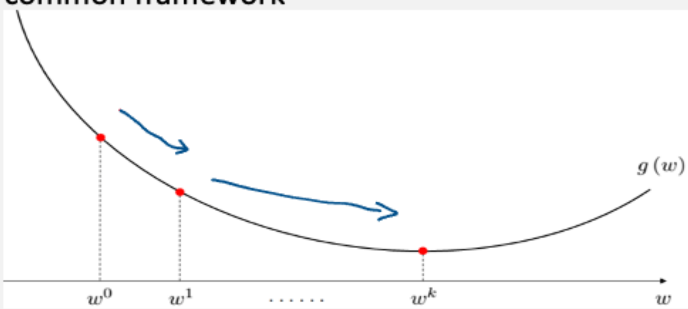
Curse of dimensionality

- The problem does not go away if we sample points randomly
- As the dimensionality n increases, smaller the fraction of samples in a n-dimensional volume

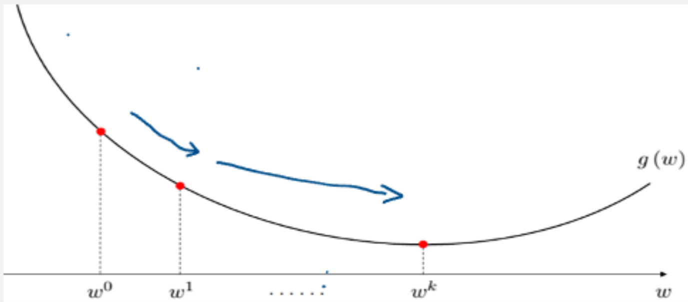


Local optimization methods

- *Local optimization methods* work by evaluating the function at a single point and sequentially updating it until an approximate minimum is reached
- Local optimization methods are by far the most popular optimization methods in machine learning
- While details vary, all local optimization methods fit into a common framework



Local optimization methods



- Starting with an initial point $\mathbf{w}^0$, local optimization methods iteratively update the current point such that:

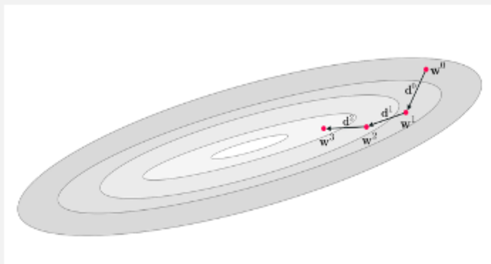
  $g\left(\mathbf{w}^0\right) > g\left(\mathbf{w}^1\right) > \cdots > g\left(\mathbf{w}^K\right)$

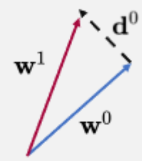
- Unlike global optimization methods, local optimization scales gracefully with the number of dimensions

Local optimization methods – a general framework

- To update the initial point $\mathbf{w}^0$ to $\mathbf{w}^1$, such that. $g(\mathbf{w}^0) > g(\mathbf{w}^1)$, a *descent direction* $\mathbf{d}^0$ is found
- Once such a descent direction is found the point is updated

$$\mathbf{w}^1 = \mathbf{w}^0 + \mathbf{d}^0$$





In general, $\mathbf{w}^k = \mathbf{w}^{k-1} + \mathbf{d}^{k-1}$ where $g(\mathbf{w}^0) > g(\mathbf{w}^1) > g(\mathbf{w}^2) > \cdots > g(\mathbf{w}^K)$

Local optimization methods

- How do we find $\mathbf{d}^{k\text{-}1}$?
- A multitude of methods exist – they differ from each other in how the descent direction is found

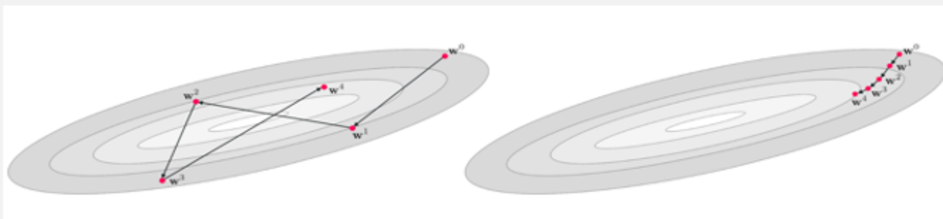
$$\mathbf{w}^k = \mathbf{w}^{k-1} + \mathbf{d}^{k-1}$$

$$\left\|\mathbf{w}^k - \mathbf{w}^{k-1}\right\|_2 = \left\| \left(\mathbf{w}^{k-1} + \mathbf{d}^{k-1}\right) - \mathbf{w}^{k-1}\right\|_2 = \left\|\mathbf{d}^{k-1}\right\|_2.$$

- $\|A\|_2$ is the 2nd norm of the vector $A = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$
- $\|A\|_2 = \sqrt[2]{\sum_{i=1}^{N} a_i^2}$
- Hence, the distance moved as a result of update is equal to the length of the vector defining the descent direction

Local optimization methods

- Depending on how our descent direction vectors are obtained, we may or may not have control over their length
  - All we ask is that they point in the right direction, 'down hill'
- Even if they point in the right direction - towards points the function value is lower - that their *length* could be problematic
  - If the length is too large, we may overshoot the minimum
  - If the length is too small, we may take forever to reach the minimum



Local optimization methods

- Most local optimization methods use a *step size parameter* (called a *learning rate* parameter in ML)

$$\mathbf{w}^k = \mathbf{w}^{k-1} + \alpha \mathbf{d}^{k-1}$$

  - In the simplest case, the step size is fixed
  - In the most general case, it can vary from step to step
- Now,

$$\left\|\mathbf{w}^k - \mathbf{w}^{k-1}\right\|_2 = \left\| \left(\mathbf{w}^{k-1} + \alpha \mathbf{d}^{k-1}\right) - \mathbf{w}^{k-1}\right\|_2 = \alpha \left\|\mathbf{d}^{k-1}\right\|_2.$$

- We can adjust the step length by choosing α

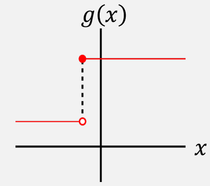
Calculus review

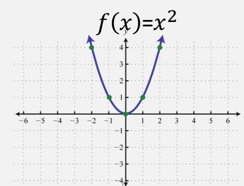
- A function of a real variable $f(x)$ is differentiable at a point a if $\lim_{\epsilon \to 0} \frac{f(a+\epsilon)-f(a)}{\epsilon}$ exists
- The limit is called the derivative of $f(x)$ at $x = a$
- The the derivative of $f(x)$ is denoted by $\frac{df}{dx}$
- If f is differentiable at a, then f must be continuous at a
  - $g(x)$ is not continuous, not differentiable
  - $f(x)$ is continuous and differentiable
  - $h(x)$ is continuous but not differentiable at $x = 0$

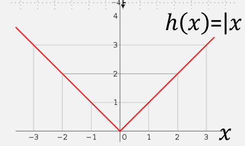
$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$







Calculus review

$$f(x) = x^2 + 3x$$

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{df}{dx} =$$

Examples

$$f(x) = x(x+3)$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{df}{dx} =$$

Examples

$$f(x) = \frac{x(x+3)}{x^2}$$

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

$$\frac{df}{dx} =$$

Partial derivatives and chain rule

Let $f(\mathbf{X}) = f(x_{0,}x_1, x_2, \ldots\ldots x_n)$

$\frac{\partial f}{\partial x_i}$ is obtained by treating all $x_i \mid i \neq j$ as constant.

Chain rule

Let $z = \varphi(u_1 \ldots . u_m)$

Let $u_i = f_i(x_{0,}x_1 \ldots\ldots x_n)$

Then $\forall k \ \frac{\partial z}{\partial x_k} = \sum_{i=1}^{m} \left(\frac{\partial z}{\partial u_i}\right)\left(\frac{\partial u_i}{\partial x_k}\right)$

- $z = f(u, v) = u^2 + 2v$
- $u = f_1(x, y) = 2x$+y
- $v = f_2(x, y) = x^2 + y$
- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$\frac{dz}{dx} = 2u(2) + 2(2x)$

$\frac{dz}{dx} = 4(2x + y) + 4x$

$\frac{dz}{dx} = 4(3x + y)$

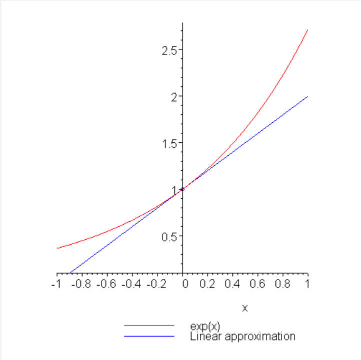
Taylor series approximation

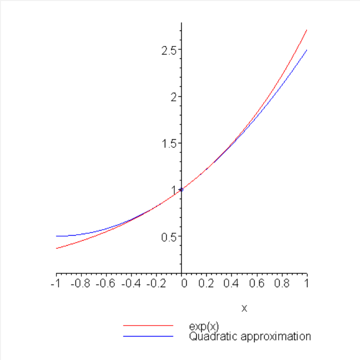
Suppose $f(x)$ is differentiable (i.e., its derivatives $\frac{df}{dx}, \frac{d^2f}{dx^2}, \cdots \frac{d^nf}{dx^n}$ exist) and $f(x)$ is continuous in the neighborhood of x_0. Then the *Taylor Series* expansion of a function $f(x)$ around $x = x_0$ is given by:

$$f(x) = f(x)\mid_{x=x_0} + \frac{df}{dx}\mid_{x=x_0}(x-x_0) + \frac{1}{2}\frac{d^2f}{dx^2}\mid_{x=x_0}(x-x_0)^2 + \ldots + \frac{1}{n!}\frac{d^nf}{dx^n}(x-x_0)$$

- Suppose $f(x) = x^2 + 1$
- $\frac{df}{dx} = 2x$
- Suppose we want to approximate f(x) at $x = 1.01$ given $f(1) = 2$
- $f(1.01) \approx f(1) + 2(1.01 - 1) \approx 2.02$
- $f(x + \Delta x) \approx f(x) + \left.\frac{df}{dx}\right|_x \Delta x$

Example


2.5
2
1.5
1
0.5
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1
x
exp(x)
Linear approximation


2.5
2
1.5
1
0.5
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1
x
exp(x)
Quadratic approximation

Taylor series approximation of multi-variable functions

The concepts introduced above extend quite naturally to the case of multivariate functions (i.e., functions of several variables). Consider a multivariate function $f(\mathbf{X}) = f(x_0, \ldots, x_n))$. Now we have *partial derivatives* that represent the rate of change of $f(\mathbf{X})$ with respect to each variable x_i. A partial derivative with respect to x_i is computed by taking the derivative of $f(x_0, \ldots x_n)$ by treating $\forall j \neq i$, x_j as though it were a constant.

Taylor Series can be used to approximate a function of several variables in a neighborhood where the function is continuous and differentiable. For example, the Taylor Series expansion for the function $\phi(x_1, x_2)$ around $\mathbf{X_0} = (x_{01}, x_{02})$ is given by:

$$\phi(\mathbf{X_0}) + \frac{\partial \phi}{\partial x_1} |_{\mathbf{X}=\mathbf{X}_0} (x_1 - x_{01}) + \frac{\partial \phi}{\partial x_2} |_{\mathbf{X}=\mathbf{X}_0} (x_2 - x_{02}) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_1^2} |_{\mathbf{X}=\mathbf{X}_0} (x_1 - x_{01})^2 + \frac{1}{2}\frac{\partial^2 \phi}{\partial x_2^2} |_{\mathbf{X}=\mathbf{X}_0} (x_2 - x_{02})^2 + \ldots$$

Taylor series approximation of multi-variable functions

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

$$f(\mathbf{x_0} + \mathbf{a}) \approx f(\mathbf{x_0}) + \nabla^T \mathbf{x}\big|_{\mathbf{x}=\mathbf{x_0}} \mathbf{a}$$

$$\nabla \mathbf{x} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_N} \end{bmatrix}$$

Taylor Series Approximation of Multivariate Functions

Let $f(\mathbf{X}) = f(x_{0,} x_1, x_2, \ldots\ldots x_n)$ be differentiable and continuous at $\mathbf{X}_0 = (x_{00,} x_{10}, x_{20}, \ldots\ldots x_{n0})$

Then

$$f(\mathbf{X}) \approx f(\mathbf{X}_0) + \sum_{i=0}^{n} \left.\left(\frac{\partial f}{\partial x}\right)\right|_{\mathbf{X}=\mathbf{X}_0} (x_i - x_{i0})$$

Minimizing functions using Gradient descent

- Suppose we want to minimize $f(z)$ with respect to z
- Suppose we start at $z = z_0$
- Suppose we want to move to $z = z_1$ such that $f(z_1) < f(z_0)$
- Change in z, $\Delta z = z_1 - z_0$
- Change in f, $\Delta f = f(z_1) - f(z_0)$
- Gradient of f at z_0 = $\left.\frac{df}{dz}\right|_{z_0} \approx \frac{\Delta f}{\Delta z}$

$$\Delta f = f(z_1) - f(z_0) = \frac{\partial f}{\partial z}|_{z=z_0} \Delta z$$

- We want $\Delta f < 0$ (f decreases as we move from $z_0$to z_1)
- We should choose. $\Delta z = -\eta \frac{\partial f}{\partial z}|_{z=z_0}$ where $\eta > 0$
- $\Delta f = -\eta \left(\left.\frac{df}{dz}\right|_{z_0}\right)^2$ is never positive (as desired)
- Hence, we must update z in the direction of the negative gradient of f

Minimizing / Maximizing Multivariate Functions

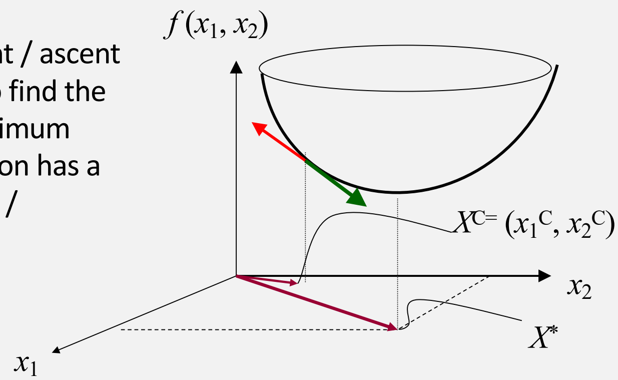
To find $\mathbf{X}^*$ that minimizes $f(\mathbf{X})$, we change current guess $\mathbf{X}^C$ in the direction of the negative gradient of $f(\mathbf{X})$ evaluated at $\mathbf{X}^C$

$$\mathbf{X}^C \leftarrow \mathbf{X}^C - \eta \left(\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1} \ldots\ldots\ldots\ldots \frac{\partial f}{\partial x_n} \right) \Bigg|_{\mathbf{X}=\mathbf{X}^C} \quad \text{(why?)}$$

for small (ideally infinitesimally small)

Minimizing / Maximizing Functions

Gradient descent / ascent is guaranteed to find the minimum / maximum when the function has a single minimum / maximum



The gradient descent algorithm

1: **input:** function g, steplength α, maximum number of steps K, and initial point $\mathbf{w}^0$
2: `for` $k = 1...K$
3: $\mathbf{w}^k = \mathbf{w}^{k-1} - \alpha \nabla g\left(\mathbf{w}^{k-1}\right)$
4: **output:** history of weights $\left\{\mathbf{w}^k\right\}_{k=0}^K$ and corresponding function evaluations $\left\{g\left(\mathbf{w}^k\right)\right\}_{k=0}^K$

- When does gradient descent stop?
- Technically (when α is chosen well) the algorithm will *halt near stationary points of a function, typically minima or saddle points*.
- How do we know this? By the very form of the gradient descent step itself.
- Say the step

$$\mathbf{w}^{k} = \mathbf{w}^{k-1} - \alpha\nabla g\left(\mathbf{w}^{k-1}\right)$$

does not move from the prior point $\mathbf{w}^{k-1}$ significantly.

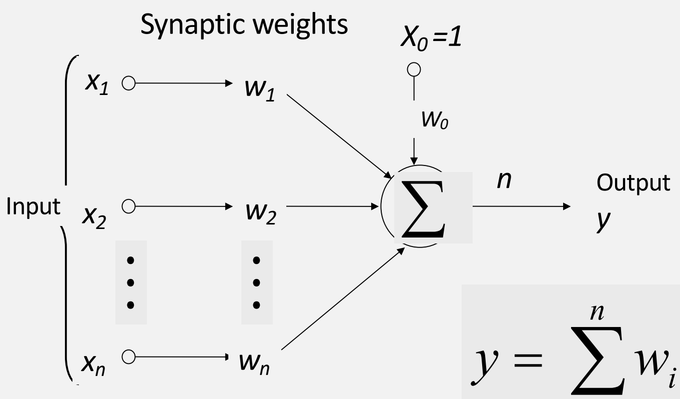
- Then this can mean only one thing: *that the direction we are traveling in is vanishing* i.e.,$-\nabla g\left(\mathbf{w}^{k}\right) \approx \mathbf{0}_{N\times 1}$
- This is - by definition - a minimum, or saddle point) of the function.

Simple Linear Regression

- In the simplest case, we have one (input), independent variable *x*, and one (output) dependent variable *y*
  - Multiple linear regression assumes an input vector **x**
  - Multivariate linear regression assumes an output vector **y**
- We will "fit" the points with a linear hyper-plane (line in the simplest case)
- Which line should we use?
  - Choose an objective function
  - For simple linear regression we choose sum squared error (SSE)
    - $\Sigma\ (d_i - y_i)^2 = \Sigma\ (e_i)^2$
  - Thus, find the line which minimizes the sum of the squared residues (e.g. least squares)

Hyperplane for multiple linear regression

Linear regression



$$y = \sum_{i=0}^{n} w_i x_i$$

Learning Task

$\mathbf{W} = [W_0 ....W_n]^T$ is the weight vector

$\mathbf{X}_p = \left[X_{0p} ....X_{np}\right]^T$ is the pth training sample

$y_p = \sum_i W_i X_{ip} = \mathbf{W} \bullet \mathbf{X}_p$ is the output of the neuron for input $\mathbf{X}_p$

$d_p = f\left(\mathbf{X}_p\right)$ is the desired output for input $\mathbf{X}_p$

$e_p = \left(d_p - y_p\right)$ is the *error* of the neuron on input $\mathbf{X}_p$

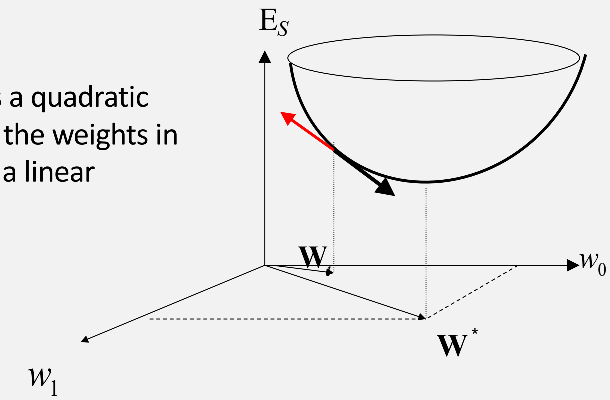
$S = \left\{\left(\mathbf{X}_p, d_p\right)\right\}$ is`a (multi) set of training examples

$E_S(\mathbf{W}) = E_S(W_0, W_1, ...W_n) = \frac{1}{2}\sum_p e_p^2$ is the estimated error of $\mathbf{W}$ on training set S

Goal: Find $\mathbf{W}^* = \underset{\mathbf{W}}{\arg\min}\, E_S(\mathbf{W})$

Learning linear functions

The error is a quadratic function of the weights in the case of a linear function



Learning linear functions

$$w_i \leftarrow w_i - \eta \frac{\partial E}{\partial w_i}$$

$$\frac{\partial E}{\partial w_i} = \frac{1}{2}\frac{\partial}{\partial w_i}\left\{\sum_p e_p^2\right\} = \frac{1}{2}\left(\sum_p \frac{\partial}{\partial w_i}\left(e_p^2\right)\right)$$

$$= \frac{1}{2}\left(\sum_p \left(2e_p\right)\frac{\partial e_p}{\partial w_i}\right) = \sum_p e_p \left(\frac{\partial e_p}{\partial y_p}\right)\left(\frac{\partial y_p}{\partial w_i}\right) = \sum_p e_p (-1)\left(\frac{\partial}{\partial w_i}\left(\sum_{j=0}^{n} w_j x_{jp}\right)\right)$$

$$= -\sum_p \left(d_p - y_p\right)\left(\frac{\partial}{\partial w_i}\left(w_i x_{ip} + \sum_{j \neq i} w_j x_{jp}\right)\right)$$

$$= -\sum_p \left(d_p - y_p\right)\left(\frac{\partial}{\partial w_i}\left(w_i x_{ip}\right) + \frac{\partial}{\partial w_i}\left(\sum_{j \neq i} w_j x_{jp}\right)\right)$$

$$= -\sum_p \left(d_p - y_p\right) x_{ip}$$

$$w_i \leftarrow w_i + \eta \sum_p \left(d_p - y_p\right) x_{ip}$$

Batch Update

$$w_i \leftarrow w_i + \eta \sum_p \left(d_p - y_p\right) x_{ip}$$

Per sample Update

$$w_i \leftarrow w_i + \eta \left(d_p - y_p\right) x_{ip}$$

General Algorithm

- Algorithm (Model algorithm for n-dimensional unconstrained minimization). Let x_k be the current estimate of x^*.
  - [Test for convergence] If the conditions for convergence are satisfied, the algorithm terminates with x_k as the solution.
  - [Compute a search direction] Compute a non-zero n-vector p_k, the direction of the search.
- Different algorithms differ primarily in their choice of the search direction

Momentum update

$$w_i(t+1) = w_i(t) + \Delta w_i(t)$$

$$\Delta w_i(t) = -\eta \left. \frac{\partial E}{\partial w_i} \right|_{w_i = w_i(t)} + \alpha \Delta w_i(t-1) \text{ where } 0 < \alpha < 1$$

$$= -\eta \sum_{\tau=0}^{t} \alpha^{t-\tau} \left. \frac{\partial E}{\partial w_i} \right|_{w_i = w_i(\tau)}$$

The momentum update allows effective learning rate to increase when feasible and decrease when necessary. Converges for $0 \leq \alpha < 1$

Locally weighted regression

Locally weighted regression involves calculating an approximation of the function value for a given input based on its nearest neighbors when needed during the approximation phase as opposed to during the learning phase.

Let the approximation be of the form

$$g(X) = w_0 + \sum_{i=1}^{N} w_i x_i$$

in a small neighborhood around a query X_q

Locally weighted regression

$$g(X) = w_0 + \sum_{i=1}^{N} w_i x_i$$

Minimize the error over the K nearest neighbors of X_q

$$E(X_q) = \frac{1}{2} \sum_{X \in KNN(X_q)} \left(f(X) - g(X)\right)^2$$

$$w_i \leftarrow w_i - \eta \frac{\partial E(X_q)}{\partial w_i}$$

$$w_i \leftarrow w_i + \eta \sum_{X \in KNN(X_q)} \left(f(X) - g(X)\right) x_i$$

Locally weighted regression

Minimize the error over all the neighbors of X_q in the training set weighted by an inverse function of distance to the neighbors

$$E_2(X_q) = \frac{1}{2} \sum_{X:(X, f(X)) \in D} \left(f(X) - g(X)\right)^2 \phi\left(d(X_q, X)\right)$$

$$w_i \leftarrow w_i - \eta \frac{\partial E_2(X_q)}{\partial w_i}$$

$$w_i \leftarrow w_i + \eta \sum_{X:(X, f(X)) \in D} \phi\left(d(X_q, X)\right)\left(f(X) - g(X)\right) x_i$$

Locally weighted regression

Minimize the error over all the neighbors of X_q in the training set weighted by an inverse function of distance over the *K* nearest neighbors

$$E_2(X_q) = \frac{1}{2} \sum_{X:(X,f(X))\in D} \left(f(X) - g(X)\right)^2 \phi\left(d(X_q, X)\right)$$

$$w_i \leftarrow w_i - \eta \frac{\partial E_2(X_q)}{\partial w_i}$$

$$w_i \leftarrow w_i + \eta \sum_{X:(X,f(X))\in D} \phi\left(d(X_q, X)\right)\left(f(X) - g(X)\right) x_i$$